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OPTIMIZATION PROBLEMS ON THRESHOLD GRAPHS

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Abstract. During the last three decades, different types of decompositions have been processed in the field of graph theory. Among these we mention: decompositions based on the additivity of some characteristics of the graph, decompositions where the adjacency law between the subsets of the partition is known, decompositions where the subgraph induced by every subset of the partition must have predeterminate properties, as well as combinations of such decompositions.

In this paper we characterize threshold graphs using the weakly decomposition, determine: density and stability number, Wiener index and Wiener polynomial for threshold graphs.

Keywords: *Threshold graph, weakly decomposition, Wiener index, Wiener polynomial.*

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1. Introduction

Threshold graphs play an important role in graph theory as well as in several applied areas such as set-packing problem (Chvátal and Hammer [4]), parallel processing (Henderson and Zalcstein [14]), allocation problems (Ordman [19]).

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally nonadjacent.

The structure of the paper is the following. In Section 2 we present the notations to be used, in Section 3 we give the notion of weakly decomposition and in Section 4 we determine the clique number, the stability number and give some applications in optimization problems.

2.General Notations

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges ([2]), having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. A vertex $z \in V$ distinguishes the non-adjacent vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say that A and B are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we simply write $N(v)$ and $N[v]$, when G appears clearly from the context. The neighbourhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number density* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertices set such that each part is a clique. $\theta(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices is a stable set o maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega(\overline{G})$ and it is called *chromatic number*.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A graph is called *cograph* if it does not contain P_4 as an induced subgraph.

A *split* graph is a graph in which the vertices can by partitioned a clique and an independent set.

Let F denote a family of graphs. A graph G is called *F-free* if none of its subgraphs is in F. The *Zykov sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

3.Preliminary results

3.1. Weakly decomposition

At first, we recall the notions of weakly component and weakly decomposition.

Definition 1. ([7], [22], [23]) *A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A.*

Definition 2. ([7], [22], [23]) *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.*

Below we remind a characterization of the weakly decomposition of a graph.

The name of "*weakly component*" is justified by the following result.

Theorem 1. ([8], [22], [23]) *Every connected and non-complete graph $G = (V, E)$ admits a weakly component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 2. ([22], [23]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weakly decomposition (A, B, C), such that $G(A)$ is a weakly component and $G(V - A) = G(B) + G(C)$.*

Theorem 2 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

Algorithm for the weakly decomposition of a graph ([22])

Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.

Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.

begin

  $A :=$ any set of vertices such that

  $A \cup N(A) \neq V$

  $N := N(A)$

  $R := V - A \cup N(A)$

  while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*

    begin

      $A := A \cup \{n\}$

      $N := (N - \{n\}) \cup (N(n) \cap R)$

      $R := R - (N(n) \cap R)$

    end

end

3.2. Threshold graphs

In this subsection we remind some results on threshold graphs.

A graph G is called *threshold* graph if $N_G(x) \subseteq N_G[y]$ or $N_G(y) \subseteq N_G[x]$ for any pair of vertices x and y in G.

Threshold graphs were first introduced by Chvátal and Hammer ([5]).

In [20], Ortiz and Villanueva-Ilufi give a structural characterization of threshold graphs for solving the following two difficult problems: enumeration of all maximal independent sets and the chromatic index problem.

Theorem 3. ([4]) A graph G is a threshold graph if and only if G does not contain a C_4, $\overline{C}_4$, P_4 as an induced subgraph.

Chvátal and Hammer also showed that threshold graphs can be recognizing in $O(n^2)$ time.

In [1], Babel showed that if G is a threshold graph then the algorithms that determine $\omega(G)$, $\chi(G)$, $\alpha(G)$ and $\theta(G)$ are $O(n+m)$ time.

Theorem 4. ([4]) A graph G is a threshold graph if and only if G is a cograph and G is a split graph.

In [6] (as well as in [12] and [17]) linear algorithms for recognizing a cograph can be found. Hammer and Simeone [13]) give an $O(n+m)$ algorithm for recognizing a split graph. Therefore, an algorithm that recognizes a threshold graph is $O(n(n+m))$.

In [18] a linear algorithm for recognizing a threshold graph can be found.

4.New results on threshold graphs

4.1. Characterization of a threshold graph using the weakly decomposition

In this paragraph we give a new characterization of threshold graphs using the weakly decomposition. Also, we determine the stability number and the clique number for threshold graphs.

Theorem 5. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a threshold graph if and only if:*
i) $A \sim N \sim R$*;*
ii) $d_G(n) = |V|-1$, $d_G(r) = |N|$, $\forall n \in N$, $\forall r \in R$*;*
iii) G(A) is threshold graph.

Proof. Let $G=(V,E)$ be a connected, uncomplete graph and (A,N,R) a weakly decomposition of G, with $G(A)$ as the weakly component.
At first, we assume that G is threshold. Then $N \sim R$ and $A \sim N$ also, as otherwise $a \in A$, $n \in N$ would exists such that $an \notin E$. Because $N = N(A)$ it follows that there exists $a_1 \in A$ such that $na_1 \in E$. As $G(A)$ is connected, a path P_{aa_1} exists. On the path from a to a_1 in P_{aa_1}, let $a_2 \in A$ the last vertex with $a_2n \notin E$ and $a_3 \in A$ the first vertex with $a_3n \in E$. Then $G(\{a_2,a_3,n,r\}) \simeq P_4$, for every $r \in R$, so i) holds.

If N would not be a clique then (as $A \sim N \sim R$) an induced C_4 would exists. This would be a contradiction, as G is threshold. So N is a clique and $A \sim N \sim R$, which leads to $d_G(n) = (|N|-1)+|A|+|R| = |V|-1$, $\forall n \in N$. So ii) also holds.
Suppose that R is not stable. Then an edge r_1r_2 ($r_1, r_2 \in R$) exists such that $G(\{r_1,r_2,a_1,a_2\}) \simeq 2K_2$, for every $a_1 \in A$ and every $a_2 \in A$, as $|A| \geq 2$. Indeed, if $|A| = 1$ then because R is not stable there exists $R' \subseteq R$ such that $G(R')$ is connected. Suppose that R' is maximal with respect to inclusion.

Then $G(R')$ is a weakly component as R' is a weakly set ($N_G(R') = N \neq A \cup N \cup (R - R') = V - R'$, $G(R')$ is connected) and R' is maximal with respect to inclusion. We have $|R'| > |A|$, contradicting the maximality of A. As $A \neq \emptyset$, it follows that $|A| \geq 2$. So R is stable.

As R is stable, $R \sim N$, $R \not\sim A$, it follows that $d_G(r) = |N|$, $\forall r \in R$. As G is threshold we have that $G(A)$ is threshold, so iii) holds, too.

Conversely, we suppose that i), ii) and iii) hold. As $A \sim N \sim R$, $A \not\sim R$ and $d_G(n) = |A| + (|N| - 1) + |R|$ it follows that N is a clique. As $R \sim N$, $R \not\sim A$ and $d_G(r) = |N|$, $\forall r \in R$ it follows that R is stable. If we suppose that $X \subset V$ exists such that $G(X) \simeq 2K_2$ then, as $A \sim N \sim R$, N clique and R stable, it follows that $X \subseteq A$, contradicting that $G(A)$ is threshold. If we suppose that $G(X) \simeq P_4$ then $X \subseteq A$, contradicting iii). In a similar manner we can prove that G is C_4-free. So G is threshold.

The above results lead to a recognition algorithm with the total execution time $O(n(n + m))$.

4.2. Determination of clique number and stability number for a threshold graph

The threshold graphs is a graph class of bounded clique-width ([3]).

Proposition 1. *If G=(V,E) is a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition with A the weakly component then*

$$\alpha(G) = max\{\alpha(G(A)) + \alpha(G(\overline{N}(A))), \alpha(G(A \cup N(A)))\}.$$

Proof. Indeed, every stable set of maximum cardinality either intersects $\overline{N}(A)$ and in this case the cardinal is $\alpha(G(A)) + \alpha(G(\overline{N}(A)))$ or it does not intersect $\overline{N}(A)$ and has the cardinal $\alpha(G(A \cup N(A)))$.

Theorem 6. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then*

$$\alpha(G) = \alpha(G(A)) + |R| \textit{ and } \omega(G) = \omega(G(A)) + |N|.$$

Proof. As R is a stable set, $A \neq \emptyset$, $G(A)$ is connected, $A \sim N$ and $A \not\sim R$ it follows that $\omega(G) = \omega(G(A)) + \omega(G(N))$. Because N is a clique it follows that $\omega(G) = \omega(G(A)) + |N|$. According to Proposition 1, $\alpha(G) = max\{\alpha(G(A)) + \alpha(G(\overline{N}(A))), \alpha(G(A \cup N(A)))\}$. Let $T \subset A \cup N(A)$ such that T is stable, with $|T| = \alpha(G(A \cup N(A)))$. As $N = N(A)$ is a clique, we have

$|T \cap N(A)| \leq 1$. If $T \cap N = \emptyset$ then $T \cup \{r\}$ is a stable set in $G(A \cup \overline{N}(A))$, $\forall r \in R = \overline{N}(A)$. If $T \cap N(A) = \{n\}$ then $(T - \{n\}) \cup \{r\}$ is a stable set in $G(A \cup \overline{N}(A))$, $\forall r \in R$. Therefore, the maximum is obtained only for the first component, that is, $\alpha(G) = \alpha(G(A)) + |R|$, because R is stable.

As a consequence of the above theorem, we give an algorithm that leads to a stable set of maximal cardinal and to a clique of maximal cardinal in a threshold graph.

Input: A threshold, connected graph with at least two nonadjacent vertices, $G = (V, E)$

Output: Determination of $\alpha(G)$ and $\omega(G)$

begin
 $S = \emptyset$; $Q = \emptyset$; $s := 0$; $q := 0$; $i := 1$; $G_i := G$;
 while $|V(G_i)| \geq 4$ *do*
  Determine a weakly decomposition (A_i, N_i, R_i) of G_i, with R_i stable, N_i clique and $G(A_i)$ threshold
  if (G_i is complete) *then*
   $S := S \cup \{v\}$, $s := s + 1$, $\forall v \in V(G_i)$;
   $Q := Q \cup V(G_i)$, $q := q + |V(G_i)|$
  else
   $S := S \cup R_i$, $s := s + |R_i|$;
   $Q := Q \cup N_i$, $q := q + |N_i|$;
   $i := i + 1$;
   $H := G_i$;
  $\alpha(G) := s + \alpha(H)$;
  $\omega(G) := q + \omega(H)$
end

Remark 1. The most time consuming operation inside the *while* loop is the determination of the decomposition (A, N, R), namely $O(n + m)$. As the *while* body executes at most n times, it follows that the total execution time is $O(n(n + m))$.

The characterization theorem of threshold graphs leads to the following result that is useful in the next section.

Corollary 2. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then if after k steps in the weakly decomposition algorithm of G we get $|A_k| \leq 3$ then $A_k \simeq K_3$ or $A_k \simeq K_2$ or $A_k \simeq K_1$.*

Proof. $G(A_k)$ is connected and $|A_k| \leq 3$. If $|A_k| = 3$ then if $G(A_k) \not\simeq K_3$ then

$G(A_k) \simeq P_3$ and we apply again the weakly decomposition algorithm with the vertex of degree 2 in N_{k+1} and the other two vertices in A_{k+1} and R_{k+1}. If $|A_k| = 2$ then $G(A_k) \simeq K_2$.

5.Some Applications in Optimization Problems

In this section we point some applications of threshold graphs in optimization problems.

Facility location analysis deals with the problem of finding optimal locations for one or more facilities in a given environment [16]. Location problems are classical optimization problems with many applications in industry and economy. The spatial location of the facilities often takes place in the context of a given transportation, communication, or transmission system. A first paradigme for location is based on the minimization of transportation cost.

According to their objective function, we can consider two types of location problems. The first type consists of those problems that use a minimax criterion. For example, if we want to determine the location of a hospital the main objective is to find a site that minimizes the maximum response time between the hospital and site of a possible emergency. More generally, the aim of the first problem type is to determine a location that minimizes the maximum distance to any other location in the network. The second type of location problems optimizes a "minimum of a sum" criterion, which is used in determining the location for a service facility like a shopping mall, for which we try to minimize the total travel time. The following centrality indices are defined in [16].

The eccentricity of a vertex u is $e_G(u) = max\{d(u, v)|v \in V\}$.

The radius is $r(G) = min\{e_G(u)|u \in V\}$.

The center of a graph G is $\mathcal{C}(G) = \{u \in V|r(G) = e_G(u)\}$.

We consider the second type of location problems. Suppose we want to place a service facility such that the total distance to all customers in the region is minimal. The problem of finding an appropriate location can be solved by computing the set of vertices with minimum total distance.

We denote the sum of the distances from a vertex u to any other vertex in a graph G=(V,E) as the total distance $s(u) = \sum_{v \in V} d(u, v)$. If the minimum total distance of G is denoted by $s(G) = min\{s(u)|u \in V\}$, the median $\mathcal{M}(G)$ of G is given by $\mathcal{M}(G) = \{u \in V|s(G) = s(u)\}$.

Our result concerning the center of a threshold graph is the following.

Theorem 7. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices. If G is threshold and if after k steps in the algorithm weakly decomposition of G we get $|A_k| \leq 3$, then the center and the median are equal to N, the radius is 1, while the excentricity is 1 for the vertices in N and 2 for the others.*

Proof. Because $A \sim N \sim R$, $A \not\sim R$, R is stable and N is a clique it follows that $e_G(u) = 1, \forall u \in N$ si $e_G(u) = 2, \forall u \in A \cup R$. So $r(G) = 1$ and $\mathcal{C}(G) = N$. It is easy to prove that

$$s_G(u) = \begin{cases} 2(|R|-1) + |N| + 2|A|, & for\ u \in R \\ (|N|-1) + |A| + |R|, & for\ u \in N \\ s_{G(A)}(u) + |N| + 2|R|, & for\ u \in R \end{cases}$$

where $s_{G(A)}(u) = |A| - 1 + \sum_{l=2}^{i-1} |R_l| (1 \leq i \leq k)$. Therefore $s(G) = s_G(u) = |V| - 1$, for $u \in N$. Therefore $\mathcal{M}(G) = N$.

The Wiener index was introduced in 1947 by Horold Wiener ([24]) and is defined as the sum of distance between all pairs of vertices in G:

$$W(G) = \sum_{u,v \in V} d_G(u, v).$$

We wish to point out that the theoretical framework is especially well elaborated for the Wiener index of trees ([9]).

The distance-counting polynomial was introduced [15] as:

$$H(G, x) = \sum_k d(G, k) x^k,$$

with $d(G, 0) = |V(G)|$ and $d(G, 1) = |E(G)|$, where $d(G, k)$ is the number of pair vertices lying at distance k to each other. This polynomial was called Wiener, by its author Hosoya, in the more recent literature [11], [21].

Theorem 8. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then: if after k steps in the algorithm weakly decomposition of G we get $|A_k| \leq 3$ then*

$$H(G, x) = [\tfrac{1}{2}(\alpha(G) - 1)^2 + |A_k|(\alpha(G) - 1)]x^2 + |E(G)|x + |V(G)| \ \textit{and}$$
$$W(G) = |E(G)| + (\alpha(G) - 1)^2 + 2|A_k|(\alpha(G) - 1).$$

Proof. According to Corollary 2, we have $\alpha(G(A_k)) = 1$ (there are no vertices in A_k at distance 2). Between every two vertices in any $R_i (i = 1, ..., k)$ the distance is 2, which means that there are $\frac{1}{2}\sum_{i=1}^{k}|R_i|(|R_i|-1)$ pairs of verices at distance 2. Between every two vertices that are placed in any $R_i, R_j(i,j=1,...,k, i\neq j)$ the distance is 2, meaning that there are $\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1,j\neq i}^{k}|R_i||R_j|$ pairs of vertices at distance 2. If a vertex is placed in $R_i(i=1,...,k)$ and the other is in A_k the distance is also 2, so there are $\sum_{i=1}^{k}|R_i||A_k|$ pairs of vertices at distance 2. As $\sum_{i=1}^{k}|R_i| = \alpha(G)-\alpha(G(A_k))$, it follows that

$$d(G,2)=\frac{1}{2}(\alpha(G)-\alpha(G(A_k))^2+|A_k|(\alpha(G)-\alpha(G(A_k)).$$

There are no vertices placed at a distance bigger than 2. Therefore

$$H(G,x)=[\frac{1}{2}(\alpha(G)-1)^2+|A_k|(\alpha(G)-1)]x^2+|E(G)|x+|V(G)|.$$

The Wiener index is the sum of all distances between all pairs of vertices, that is $\sum_{p=0}^{2}pd(G,p)$, because there are no vertices at a distance bigger than 2. Therefore, $W(G)=0\cdot|V(G)|+1\cdot|E(G)|+2\cdot d(G,2)$=$|E(G)|+(\alpha(G)-1)^2+2|A_k|(\alpha(G)-1)$.

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