A CLASS OF OPTIMAL QUADRATURE FORMULAE

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Abstract. The optimal quadrature formulae in the sense of minimal error bounds are obtained. The corrected quadrature rules of these quadrature formulae are considered.

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1. Introducere

The problem to construct the optimal quadrature formulas was studied by many authors ([1], [2], [8], [9], [10], [11], [13]). In recent years N. Ujević [8]-[10] used a new approach for obtaining optimal two-point and three-point quadrature formulae. Further, some error inequalities have also been obtained by Ujević for different classes of functions.

In [13], F. Zafar, N.A.Mir present some improvements an generalizations of Ujević's results. It is obtained the following family of four-point quadrature rule.

$$\int_{-1}^{1} f(t)dt = hf(-1) + (1-h)f(x) + (1-h)f(y) + hf(1) + \mathcal{R}[f], \text{ where}$$

$$\mathcal{R}[f] = \int_{-1}^{1} K(x,t)f''(t)dt,$$

$$K(x,t) = \begin{cases} \frac{1}{2}(t+1-h)^2 - \frac{1}{2}h^2, & t \in [-1,x], \\ \frac{1}{2}t^2 + (1-h)x - h + \frac{1}{2}, & t \in (x,-x), \\ \frac{1}{2}(t-1+h)^2 - \frac{1}{2}h^2, & t \in [-x,1], \end{cases}$$

for $x \in [-1+2h, 0]$, $h \in [0, 1/2]$.

The remainder term can be evaluated in the following way

$$|\mathcal{R}[f]| \leq \|f''\|_{\infty} \int_{-1}^{1} |K(x,t)|\, dt. \tag{1}$$

Putting condition to find an x that minimizes $\displaystyle\int_{-1}^{1} |K(x,t)|\, dt$ it is obtained the bellow result

Theorem 1 *[13] Let $I \subset \mathbf{R}$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to \mathbf{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then,*

$$\begin{aligned} \int_{-1}^{1} f(t)dt &= \Big[hf(-1) + (1-h)f(-4+4h+2\sqrt{3-6h+4h^2}) \\ &+ (1-h)f(4-4h-2\sqrt{3-6h+4h^2}) + hf(1)\Big] + \mathcal{R}[f], \end{aligned} \tag{2}$$

where $|\mathcal{R}[f]| \leq 2\Delta(h)\, \|f''\|_{\infty}$, $h \in \left[0, \dfrac{1}{2}\right]$, *and* $\Delta(h)$ *is defined as*

$$\begin{aligned} \Delta(h) = &\frac{52}{3}h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(1-h)^2\sqrt{4h^2-6h+3} \\ &+ \frac{2}{3}\left[8h^2 - 14h + 7 - 4(1-h)\sqrt{4h^2-6h+3}\right]^{\frac{3}{2}}. \end{aligned}$$

In this paper the estimation of remainder term is given in terms of variety of norms, from an inequality point of view. Further, we will consider so called perturbed (corrected) quadrature rules. The estimate of the error in corrected rule is better then in the original rule, in generally. In recent years the corrected quadrature rules have been in the attention of many authors (see [3], [4], [5], [6], [7], [12]).

2. The optimal quadrature formula

We know that the remainder term of quadrature formula (1) may be evaluated as

$$|\mathcal{R}[f]| \leq \|f''\|_2 \left\{ \int_{-1}^{1} [K(x,t)]^2\, dt \right\}^{\frac{1}{2}}. \tag{3}$$

For the quadrature formula (2) obtained by F. Zafar and N.A. Mir we give the following estimation of the remainder term

$$|\mathcal{R}[f]| \leq \omega(h)\|f''\|_2, \text{ where } \omega(h) = \left\{ \int_{-1}^{1} [K(x,t)]^2\, dt \right\}^{\frac{1}{2}}.$$

For $x=-4+4h+2\sqrt{3-6h+4h^2}$ we have

$$\int_{-1}^{1}[K(x,t)]^2\,dt=\int_{-1}^{x}\left[\frac{1}{2}(t+1-h)^2-\frac{1}{2}h^2\right]^2dt$$

$$+\int_{x}^{-x}\left[\frac{1}{2}t^2+(1-h)x-h+\frac{1}{2}\right]^2dt+\int_{-x}^{1}\left[\frac{1}{2}(t-1+h)^2-\frac{1}{2}h^2\right]^2dt.$$

Calculating the above integrals we obtain

$$\omega(h)=\left\{\left(\frac{16}{3}-\frac{64}{3}h+\frac{80}{3}h^2-\frac{32}{3}h^3\right)\sqrt{3-6h+4h^2}\right.$$

$$\left.+\frac{277}{6}h-\frac{254}{3}h^2+\frac{208}{3}h^3-\frac{64}{3}h^4-\frac{277}{30}\right\}^{\frac{1}{2}}.$$

The main aim of this section is to present a minimal estimation of the error bound (3). If we require to find an x that minimizes $\int_{-1}^{1}[K(x,t)]^2\,dt$ we obtain the following result.

Theorem 2 *Let $I\subset\mathbf{R}$ be an open interval such that $[-1,1]\subset I$ and let $f:I\to\mathbf{R}$ be a twice differentiable function such that f'' is integrable. Then,*

$$\begin{aligned}\int_{-1}^{1}f(t)dt &= hf(-1)+(1-h)f(-3h+3+\sqrt{9h^2-12h+6}) \\ &+ (1-h)f(3h-3+\sqrt{9h^2-12h+6})+hf(1)+\mathcal{R}[f],\end{aligned}\tag{4}$$

where $|\mathcal{R}[f]|\le\Phi(h)\,\|f''\|_2$, $h\in\left(-\infty,\frac{1}{2}\right]$, and $\Phi(h)$ is defined as

$$\Phi(h)=\left\{-36h^5+144h^4-240h^3+\frac{632}{3}h^2-98h+\frac{98}{5}\right.$$

$$\left.+\left[-12h^4+40h^3-52h^2+32h-8\right]\sqrt{9h^2-12h+6}\right\}^{\frac{1}{2}}.$$

Proof. We define

$$F(x)=\int_{-1}^{1}[K(x,t)]^2\,dt=\int_{-1}^{x}\left[\frac{1}{2}(t+1-h)^2-\frac{1}{2}h^2\right]^2dt$$

$$+\int_{x}^{-x}\left[\frac{1}{2}t^2+(1-h)x-h+\frac{1}{2}\right]^2dt+\int_{-x}^{1}\left[\frac{1}{2}(t-1+h)^2-\frac{1}{2}h^2\right]^2dt$$

$$=\frac{1}{6}(h-1)x^4+\frac{4}{3}(2h-h^2-1)x^3+(3h-1-2h^2)x^2-\frac{1}{2}h+\frac{2}{3}h^2+\frac{1}{10}.$$

We find that $x^*=3h-3+\sqrt{9h^2-12h+6}$ is the global minima of F.

Remark 1 *In the bellow figure we give a graphical representation of the functions ω, respectively Φ which appear in L_2-norm estimation of the remainder term of quadrature formulas (2), respectively (4).*

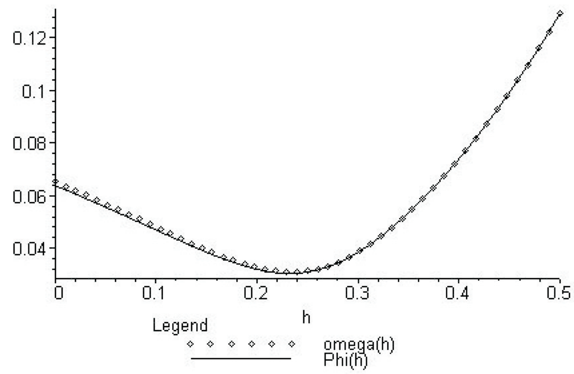

Figure 1.

Remark 2 *We recapture the Gauss two-point quadrature formula for $h = 0$ and $x = -\frac{\sqrt{3}}{3}$, namely*

$$\int_{-1}^{1} f(t)dt = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) + \mathcal{R}[f], \text{ where}$$

$$|\mathcal{R}[f]| \leq \sqrt{-\frac{34}{135} + \frac{4}{27}\sqrt{3}}\|f''\|_2 \approx 0.0689\|f''\|_2.$$

Remark 3 *For $h = 1/6$ and $x = -\frac{\sqrt{5}}{5}$, we get Lobatto four-point quadrature rule as follows:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{6}\left[f(-1) + 5f\left(-\frac{\sqrt{5}}{5}\right) + 5f\left(\frac{\sqrt{5}}{5}\right) + f(1)\right] + \mathcal{R}[f], \text{ where}$$

$$|\mathcal{R}[f]| \leq \sqrt{-\frac{11}{135} + \frac{1}{27}\sqrt{5}}\|f''\|_2 \approx 0.0365\|f''\|_2.$$

Remark 4 *For $h=1/4$ and $x=-\dfrac{1}{3}$, we get 3/8 Simpson's rule as follows:*

$$\int_{-1}^{1} f(t)dt = \frac{1}{4}\left[f(-1)+3f\left(-\frac{1}{3}\right)+3f\left(\frac{1}{3}\right)+f(1)\right]+\mathcal{R}[f],\ \textit{where}$$

$$|\mathcal{R}[f]| \le \frac{1}{\sqrt{810}}\|f''\|_2 \approx 0.0351\|f''\|_2.$$

By considering the problem on the interval $[a,b]$, the following result is obtained

Theorem 3 *Let $I\subset \mathbf{R}$ be an open interval such that $[a,b]\subset I$ and let $f: I\to\mathbf{R}$ be a twice-differentiable function such that f'' is integrable. Then*

$$\int_a^b f(t)dt = \frac{b-a}{2}\left[hf(a)+(1-h)f\left(\frac{a+b}{2}-\frac{b-a}{2}x^*\right)\right.$$
$$\left.+(1-h)f\left(\frac{a+b}{2}+\frac{b-a}{2}x^*\right)+hf(b)\right]+\mathcal{R}[f],$$

where

$$x^* = 3h-3+\sqrt{9h^2-12h+6},$$
$$|\mathcal{R}[f]| \le \frac{(b-a)^3}{8}\Phi(h)\|f''\|_2,$$

$h\in\left(-\infty,\dfrac{1}{2}\right]$, *and $\Phi(h)$ is as defined above.*

Theorem 4 *Let $I\subset\mathbf{R}$ be an open interval such that $[-1,1]\subset I$ and let $f: I\to\mathbf{R}$ be a twice differentiable function such that $f''(t)$ is bounded and integrable. Then,*

$$\begin{aligned}\int_{-1}^{1} f(t)dt &= hf(-1)+(1-h)f(-3h+3+\sqrt{9h^2-12h+6}) \\ &+ (1-h)f(3h-3+\sqrt{9h^2-12h+6})+hf(1)+\mathcal{R}[f],\end{aligned} \tag{5}$$

where $|\mathcal{R}[f]|\le\Psi(h)\left\|f''\right\|_\infty$, $h\in\left[0,\dfrac{1}{2}\right]$, *and* $\Psi(h)$ *is defined as*

$$\begin{aligned}\Psi(h) =& \frac{1}{3}-(1-h)\left(3h-3+\sqrt{9h^2-12h+6}\right)^2 \\ &-\frac{8}{3}(1-h)\sqrt{2h-1-2(1-h)\left(3h-3+\sqrt{9h^2-12h+6}\right)}\left(3h-3+\sqrt{9h^2-12h+6}\right) \\ &+\frac{8}{3}\left(h-\frac{1}{2}\right)\sqrt{2h-1-2(1-h)\left(3h-3+\sqrt{9h^2-12h+6}\right)}+\frac{8}{3}h^3-h.\end{aligned}$$

3. The coerrected quadrature formula

By corrected quadrature rule we mean the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points.

The main aim of this section is to give the corrected quadrature formula of (4). Also we will show that the estimate of the error in corrected rule is better then in the original rule.

We define

$$A := 9h^3 - 24h^2 + 22h - \frac{22}{3} + 3(h-1)^2\sqrt{9h^2-12h+6},$$

$$\tilde{K}(x^*,t) = K(x^*,t) - A, \text{ where } x^* = 3h - 3 + \sqrt{9h^2-12h+6}.$$

The corrected quadrature formula of (4) is defined bellow:

$$\int_{-1}^{1} f(x)dx = hf(-1) + (1-h)f(x^*) + (1-h)f(-x^*) + hf(1)$$
$$+A\left[f'(1) - f'(-1)\right] + \tilde{\mathcal{R}}[f],$$

where $\tilde{\mathcal{R}}[f] = \displaystyle\int_{-1}^{1} \tilde{K}(x^*,t)f''(t)dt$.

The remainder term $\tilde{\mathcal{R}}[f]$ can be evaluated in the following way

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \left\{\int_{-1}^{1}\left[\tilde{K}(x^*,t)\right]^2 dt\right\}^{\frac{1}{2}} \cdot \|f''\|_2 = \Omega(h)\|f''\|_2,$$

where

$$\Omega(h) = \left\{-\frac{8818}{45} - 324h^6 - 3744h^4 + 1692h^5 + \frac{3586}{3}h - \frac{9406}{3}h^2 + 4512h^3\right.$$
$$\left. + \left(-108h^5 + 80 - 408h + 852h^2 - 908h^3 + 492h^4\right)\sqrt{9h^2-12h+6}\right\}^{\frac{1}{2}}.$$

Can be proved easy that $\Omega(h) \leq \Phi(h)$, $h \in (-\infty, 1/2]$, namely, the error in the corrected rule is better then in the original rule. In the Figure 2 are represented the functions Ω, Φ which depends on h.

Let $f, g : [a,b] \to \mathbf{R}$ be integrable functions on $[a,b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \qquad (6)$$

is well known in the literature as the Čebyšev functional and the inequality $|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$ holds. Denote by $\sigma(f,a,b) = \sqrt{(b-a)T(f,f)}$.

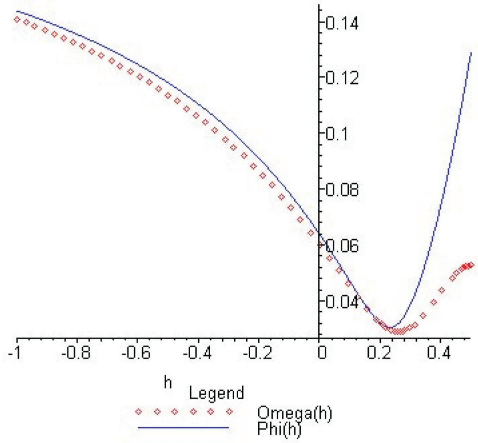

Figure 2.

Theorem 5 *Let $I \subset \mathbf{R}$ be an open interval such that $[-1,1] \subset I$ and let $f : I \to \mathbf{R}$ be a twice differentiable function such that f'' is integrable. Then,*

$$|\tilde{\mathcal{R}}[f]| \leq \Omega(h)\sigma(f'',-1,1).$$

Proof. We have

$$\tilde{\mathcal{R}}[f] = \int_{-1}^{1} \tilde{K}(x^*,t)f''(t)dt = \int_{-1}^{1} \left[K(x^*,t) - \frac{1}{2}\int_{-1}^{1} K(x^*,t)dt\right] f''(t)dt$$

$$= \int_{-1}^{1} K(x^*,t)f''(t)dt - \frac{1}{2}\int_{-1}^{1} K(x^*,t)dt \int_{-1}^{1} f''(t)dt = 2T(K,f'').$$

Therefore,

$$|\tilde{\mathcal{R}}[f]| \leq \sqrt{2T(K,K)} \cdot \sqrt{2T(f'',f'')} = \Omega(h)\sigma(f'',-1,1).$$

In a similar way the corrected quadrature formula of (2) is defined bellow:

$$\int_{-1}^{1} f(x)dx = hf(-1) + (1-h)f(x^*) + (1-h)f(-x^*) + hf(1)$$

$$+B\left[f'(1) - f'(-1)\right] + \tilde{\mathcal{R}}[f],$$

where

$$\tilde{\mathcal{R}}[f] = \int_{-1}^{1} \tilde{K}(x^*,t)f''(t)dt,$$

$$\tilde{K}(x^*,t) = K(x^*,t) - B, \text{ with } x^* = -4 + 4h + 2\sqrt{3-6h+4h^2},$$
$$B = 16h^3 - 44h^2 + \frac{83}{2}h - \frac{83}{6} + 8(h-1)^2\sqrt{3-6h+4h^2}.$$

The remainder term $\tilde{\mathcal{R}}[f]$ can be evaluated in the following way

$$\left|\tilde{\mathcal{R}}[f]\right| \leq \left\{\int_{-1}^{1}\left[\tilde{K}(x^*,t)\right]^2 dt\right\}^{\frac{1}{2}} \cdot \|f''\|_2 = \eta(h)\|f''\|_2,$$

where

$$\eta(h) = \left\{-1024h^6 + 5632h^5 + \frac{9293}{2}h - \frac{71111}{6}h^2 + \frac{49352}{3}h^3 - \frac{39232}{3}h^4 - \frac{34918}{45}\right.$$
$$\left. + \left(448 + \frac{13600}{3}h^2 - \frac{6704}{3}h - \frac{14000}{3}h^3 - 512h^5 + 2432h^4\right)\sqrt{3-6h+4h^2}\right\}^{\frac{1}{2}}.$$

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