RESULTS ON THE CONVERGENCE OF THE WEAK SOLUTION FOR THE STATIC PROBLEM OF PERIODIC ELASTIC MEDIA PROBLEMS

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Abstract. The static equations governing the stress state and the deformation of a periodic medium are established based on the homogenization method. Properties of the stress and elasticity tensors are obtained. The weak convergence of the solution of the microscopic problem is proven.

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1. The phisical problem: homogenized equations

When the material has a strong heterogeneous structure, it is practically impossible to determine its properties in each point. As a result, the porous media study is performed by approximation with periodic media which allow to the microscopic equations to had through homogenization to macroscopic laws and in certain conditions the solution of the microscopic problem is convergent to the solution of the macroscopic one.

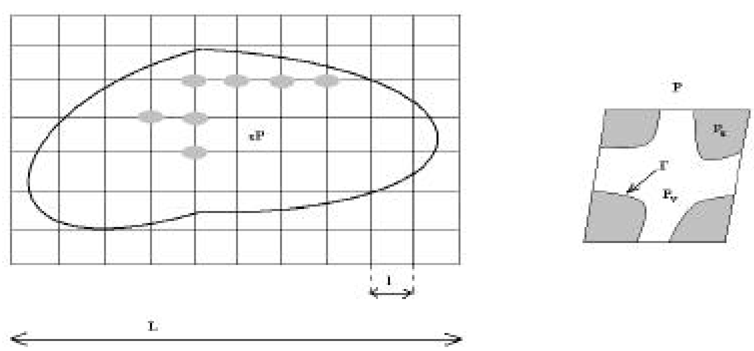
In [5] Sanchez-Palencia applies the homogenization method to the case of composite materials. He investigates static problems such as: the thermal conduction problem in a periodic medium with Dirichlet and Neumann type boundary conditions, the elasticity problem with Dirichlet type boundary conditions. For the thermal conduction equation he proved a weak convergence of the weak solution of the microscopic problem to the weak solution of the macroscopic one.

Based on Sanchez-Palencia techniques, for the linear elasticity static problem, the macroscopic equations for porous media are established and the weak solution convergence of the microscopic problem toward the weak solution of the macroscopic one is investigated.

This procedure implies that the medium is considered as having a periodic structures.

The homogenization method in elasticity problems is used also by Abdulle [1] for a polygonal domain with the numerical investigation of the homogenized equations based on the finite elaments method (FEM).

Let us consider a porous body which occupies the region $\Omega \subset \mathbf{R}^3$ (having the characteristic dimension L) with a piecewise smooth boundary $\partial\Omega$. The coordinates of a certain point in the domain $\overline{x} = (x_1, x_2, x_3) \in \Omega$ are written with respect to the well known coordinate system $\mathcal{R} = \{O; \overline{e}_1, \overline{e}_2, \overline{e}_3\}$. Following [3], [5], Ω is assumed to be periodically domain. The microstructure is defined by periodic cell P (having the characteristic dimension l) with a solid part P_S and a empty part P_V. For example, see the next figure:



- Nomenclature:

(a) the porous medium is enclosed in $\Omega = (0,\ L)^3$;

(b) the microstructure is defined by $\omega^l = (0,\ l)^3$ with a solid part ω^l_S ($\overline{\omega^l_S} \subset \omega^l$) and a empty part $\omega^l_V = \omega^l - \omega^l_s$;

(c) the cube Ω has N cells (ω^l), i.e., $\Omega = \bigcup_{j=1}^{N} \omega^l_j$, where $\omega^l_j = x_j + \omega^l$, $N = \frac{L^3}{l^3} \in \mathbf{N}$, $\omega^l_{S_j} = x_j + \omega^l_S$ and $\omega^l_{V_j} = x_j + \omega^l_V$;

(d) denote $\epsilon = \frac{l}{L} = \frac{1}{\sqrt[3]{N}} << 1$,

$$\Omega^\epsilon = \bigcup_{j=1}^{N} \omega^l_{S_j}, \quad \Gamma^\epsilon = \bigcup_{j=1}^{N} \partial\omega^l_{S_j};$$

We have $\partial\Omega^\epsilon = \Gamma^\epsilon \bigcup \partial\Omega$;

(e) The basic cell is $Y = \frac{1}{\epsilon}\omega^l = (0,\ L)^3$ with the solid part $Y_S = \frac{1}{\epsilon}\omega^l_S$ and the empty part $Y_V = \frac{1}{\epsilon}\omega^l_V$;

(f) In dimensionless variables $x'_i = \frac{x_i}{L}$, $i = 1, 2, 3$, we have:

$$\Omega \to \Omega' = \frac{1}{L}\Omega = (0,1)^3, \quad \Omega^\epsilon \to \Omega'^\epsilon = \frac{1}{L}\Omega^\epsilon, \quad \Gamma^\epsilon \to \Gamma'^\epsilon = \frac{1}{L}\Gamma^\epsilon,$$

$$\omega^l \to \omega'^\epsilon = \frac{1}{L}\omega^l = (0,\epsilon)^3, \quad \omega^l_S \to \omega'^\epsilon_S = \frac{1}{L}\omega^l_S, \quad \omega^l_V \to \omega'^\epsilon_V = \frac{1}{L}\omega^l_V,$$

$$Y \to Y' = \frac{1}{L}Y = (0,1)^3, \quad Y_S \to Y'_S = \frac{1}{L}Y_S, \quad Y_V \to Y'_V = \frac{1}{L}Y_V;$$

(g) The following relations are valid:

$\Omega' = \bigcup_{j=1}^{N} \omega'^\epsilon_j$, where $\omega'^\epsilon_j = \frac{1}{L}x_j + \omega'^\epsilon$ and $\begin{cases} \omega'^\epsilon_{S_j} = \frac{1}{L}x_j + \omega'^\epsilon_S \\ \omega'^\epsilon_{V_j} = \frac{1}{L}x_j + \omega'^\epsilon_V \end{cases}$,

$\Omega'^\epsilon = \bigcup_{j=1}^{N} \omega'^\epsilon_{S_j}$, $\Gamma'^\epsilon = \bigcup_{j=1}^{N} \partial\omega'^\epsilon_{S_j}$, where

$\partial\omega'^\epsilon_{S_j} = \frac{1}{L}x_j + \partial\omega'^\epsilon_S$, $\partial\Omega'^\epsilon = \Gamma'^\epsilon \cup \partial\Omega'$, $\partial Y'_S = \frac{1}{\epsilon}\partial\omega'^\epsilon_S$;

(h) The porosity is denoted by $\Pi_Y = |Y'_V|$.

The dimensionless microscopic problem is written:

$$\frac{1}{L}\frac{\partial}{\partial x_j}[\frac{1}{L}a^\epsilon_{ijkl}e_{kl}(\overline{u}'^\epsilon)] = -\rho^\epsilon_0 f'_i(\overline{x}') \ \text{ in } \Omega'^\epsilon, \quad i = 1, 2, 3. \tag{1}$$

Let us assume the asymptotic expansion:

$$\overline{u}'^\epsilon = \overline{u}'^0(\overline{x}', \overline{y}') + \epsilon\overline{u}'^1(\overline{x}', \overline{y}') + \epsilon^2\overline{u}'^2(\overline{x}', \overline{y}') + ..., \tag{2}$$

with $\overline{y}' = \frac{\overline{x}'}{\epsilon} \in Y'_S$, $\overline{x}' \in \Omega'$, and the functions $\overline{u}'^0$, $\overline{u}'^1$, ... are periodic functions in $\overline{y}'$.

The small deformations tensor can be written:

$$e_{kl}(\overline{u}'^{\epsilon}) = \frac{1}{2}\Big(\frac{\partial u'^{\epsilon}_k}{\partial x'_l} + \frac{\partial u'^{\epsilon}_l}{\partial x'_k}\Big) = \frac{1}{2}\Big(\frac{\partial u'^{0}_k}{\partial x'_l} + \frac{\partial u'^{0}_l}{\partial x'_k}\Big) + \frac{1}{\epsilon}\frac{1}{2}\Big(\frac{\partial u'^{0}_k}{\partial y'_l} + \frac{\partial u'^{0}_l}{\partial y'_k}\Big) +$$

$$+\epsilon\frac{1}{2}\Big(\frac{\partial u'^{1}_k}{\partial x'_l} + \frac{\partial u'^{1}_l}{\partial x'_k}\Big) + \frac{1}{2}\Big(\frac{\partial u'^{1}_k}{\partial y'_l} + \frac{\partial u'^{1}_l}{\partial y'_k}\Big) + \epsilon^2\frac{1}{2}\Big(\frac{\partial u'^{2}_k}{\partial x'_l} + \frac{\partial u'^{2}_l}{\partial x'_k}\Big) + \epsilon\frac{1}{2}\Big(\frac{\partial u'^{2}_k}{\partial y'_l} + \frac{\partial u'^{2}_l}{\partial y'_k}\Big) + ... =$$

$$= \frac{1}{\epsilon}\frac{1}{2}\Big(\frac{\partial u'^{0}_k}{\partial y'_l} + \frac{\partial u'^{0}_l}{\partial y'_k}\Big) + e^0_{kl}(\overline{x}', \overline{y}') + \epsilon e^1_{kl}(\overline{x}', \overline{y}') + ... \tag{3}$$

where the folowing notation were introduced:

$$\begin{cases} e^0_{kl}(\overline{x}', \overline{y}') = e_{kl\overline{x}'}(\overline{u}'^0) + e_{kl\overline{y}'}(\overline{u}'^1) \\ e^1_{kl}(\overline{x}', \overline{y}') = e_{kl\overline{x}'}(\overline{u}'^1) + e_{kl\overline{y}'}(\overline{u}'^2) \\ e^2_{kl}(\overline{x}', \overline{y}') = e_{kl\overline{x}'}(\overline{u}'^2) + e_{kl\overline{y}'}(\overline{u}'^3) \\ \text{...} \end{cases} \tag{4}$$

with

$$\begin{cases} e_{kl\overline{x}'}(\overline{v}) = \frac{1}{2}\Big(\frac{\partial v_k}{\partial x'_l} + \frac{\partial v_l}{\partial x'_k}\Big) \\ e_{kl\overline{y}'}(\overline{v}) = \frac{1}{2}\Big(\frac{\partial v_k}{\partial y'_l} + \frac{\partial v_l}{\partial y'_k}\Big) \end{cases}, \qquad \forall\ \overline{v} = \overline{v}(\overline{x}', \overline{y}'). \tag{5}$$

With (4) the small deformation tensor $(e_{ij}(\overline{u}'^{\epsilon}))^3_{i,j=1}$ has the form:

$$e_{ij}(\overline{u}'^{\epsilon}) = \frac{1}{\epsilon}e_{ij\overline{y}'}(\overline{u}'^0) + e^0_{ij}(\overline{x}', \overline{y}') + \epsilon e^1_{ij}(\overline{x}', \overline{y}') + ..., \quad i, j = 1, 2, 3, \tag{6}$$

leading to the stress tensor $(\sigma^{\epsilon}_{ij})^3_{i,j=1}$:

$$\sigma^{\epsilon}_{ij} = a^{\epsilon}_{ijkl}\Big[\frac{1}{\epsilon}e_{kl\overline{y}'}(\overline{u}'^0) + e^0_{ij}(\overline{x}', \overline{y}') + \epsilon e^1_{ij}(\overline{x}', \overline{y}') + \epsilon^2 e^2_{ij}(\overline{x}', \overline{y}') + ...\Big] =$$

$$= \frac{1}{\epsilon}a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^0) + \sigma^0_{ij}(\overline{x}', \overline{y}') + \epsilon\sigma^1_{ij}(\overline{x}', \overline{y}') + \epsilon^2\sigma^2_{ij}(\overline{x}', \overline{y}') + ... \tag{7}$$

where

$$\begin{cases} \sigma^0_{ij}(\overline{x}', \overline{y}') = a^{\epsilon}_{ijkl}e^0_{kl}(\overline{x}', \overline{y}') \\ \sigma^1_{ij}(\overline{x}', \overline{y}') = a^{\epsilon}_{ijkl}e^1_{kl}(\overline{x}', \overline{y}') \\ \text{.................................} \end{cases}. \tag{8}$$

Let us assume the operator $\frac{d}{dx'_i} = \frac{\partial}{\partial x'_i} + \frac{1}{\epsilon}\frac{\partial}{\partial y'_i}$, $i = 1, 2, 3$.

Replacing (7) in (1) we get:

$$\begin{aligned} &\tfrac{1}{L^2}\Big(\tfrac{\partial}{\partial x'_j} + \tfrac{1}{\epsilon}\tfrac{\partial}{\partial y'_j}\Big)\Big(\tfrac{1}{\epsilon}a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^0) + \sigma^0_{ij}(\overline{x}',\overline{y}') + \epsilon\sigma^1_{ij}(\overline{x}',\overline{y}') + \\ &\quad +\epsilon^2\sigma^2_{ij}(\overline{x}',\overline{y}') + ...\Big) = -\rho_0 f'_i, \ \text{ in } \Omega'^{\epsilon},\ i = 1, 2, 3. \end{aligned} \tag{9}$$

In (9) considering the coefficients of ϵ^{-i}, $i = 2, 1, 0$, we have:

$$\frac{\partial}{\partial y'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^0)\Big) = 0, \quad \text{în } Y'_S,\ i = 1, 2, 3, \tag{10}$$

$$\frac{\partial}{\partial x'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^0)\Big) + \frac{\partial}{\partial y'_j}\Big(\sigma^0_{ij}(\overline{x}',\overline{y}')\Big) = 0, \quad \text{în } Y'_S,\ i = 1, 2, 3, \tag{11}$$

$$\frac{1}{L^2}\frac{\partial}{\partial x'_j}\Big(\sigma^0_{ij}(\overline{x}',\overline{y}')\Big) + \frac{1}{L^2}\frac{\partial}{\partial y'_j}\Big(\sigma^1_{ij}(\overline{x}',\overline{y}')\Big) = -\rho_0 f'_i, \ \text{ în } Y'_S,\ i = 1, 2, 3. \tag{12}$$

Relation (10) shows the dependence of $\overline{u}'^0$ only on the macroscopic variable $\overline{x}'$.

Equations (11), (12) can be rewritten:

$$\begin{cases} \frac{\partial}{\partial x'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^0)\Big) + \frac{\partial}{\partial y'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{x}'}(\overline{u}'^0) + a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^1)\Big) = 0, \quad \text{în } Y'_S, \\ \overline{u}'^1 \quad Y' - \text{periodică} \\ \dfrac{1}{|Y'|}\displaystyle\int_{Y'_S}\overline{u}'^1\, d\overline{y}' = 0 \end{cases} \quad (i = 1, 2, 3) \tag{13}$$

and

$$\frac{1}{L^2}\frac{\partial}{\partial x'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{x}'}(\overline{u}'^0) + a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^1)\Big) + \frac{1}{L^2}\frac{\partial}{\partial y'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{x}'}(\overline{u}'^1) + a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^2)\Big) = -\rho_0 f'_i, \tag{14}$$

in $Y'_S, i = 1, 2, 3$, respectively.

The homogenized elasticity tensor can be evaluated from the simplified form of (13):

$$\begin{cases} \frac{\partial}{\partial y'_j}\Big(a^{\epsilon}_{ijkl}e_{kl\overline{y}'}(\overline{u}'^1)\Big) = -\frac{\partial a^{\epsilon}_{ijkl}}{\partial y'_j}e_{kl\overline{x}'}(\overline{u}'^0), \quad \text{in } Y'_S \\ \overline{u}'^1 \quad Y' - \text{periodic} \\ \dfrac{1}{|Y'|}\displaystyle\int_{Y'_S}\overline{u}'^1\, d\overline{y}' = 0 \end{cases} \quad (i = 1, 2, 3). \tag{15}$$

Based on the liniarity of $(15)_1$ we seak for a solution of the form :

$$\overline{u}'^1 = \overline{w}^{mh} e_{mh\overline{x}'}(\overline{u}'^0), \text{ în } Y'_S, \tag{16}$$

with $\overline{w}^{mh}$ Y'- periodic functions. Let use assume fixed $m, h \in \{1, 2, 3\}$ such that $e_{mh\overline{x}'}(\overline{u}'^0) = 1$. We have (15) and the variational formulation:

$$\begin{cases} \text{Find } \overline{w}^{mh} \in \widetilde{H}^1_{per}(Y') \text{ such that} \\ \displaystyle\int_{Y'_S} a^\epsilon_{ijkl} e_{kl\overline{y}'}(\overline{w}^{mh}) \cdot e_{ij\overline{y}'}(\overline{v})\ d\overline{y}' = \int_{Y'_S} \frac{\partial a^\epsilon_{ijmh}}{\partial y'_j} \cdot v_i\ d\overline{y}', \quad \forall \overline{v} \in \widetilde{H}^1_{per}(Y') \end{cases} (i = 1, 2, 3). \tag{17}$$

From relations $(8)_1$, $(4)_1$ and (16) we obtain:

$$\sigma^0_{ij}(\overline{x}', \overline{y}') = a^\epsilon_{ijkl}\Big[\delta_{km}\delta_{lh} + e_{kl\overline{y}'}(\overline{w}^{mh})\Big] e_{mh\overline{x}'}(\overline{u}'^0),$$

in which mediation relative to the Y' cell, we obtain the relation:

$$\begin{aligned} \widetilde{\sigma}^0_{ij}(\overline{x}') &= \frac{1}{|Y'|}\int_{Y'_S} \sigma^0_{ij}(\overline{x}', \overline{y}')\ d\overline{y}' = \\ &= \Big(\frac{1}{|Y'|}\int_{Y'_S} a^\epsilon_{ijkl}\Big[\delta_{km}\delta_{lh} + e_{kl\overline{y}'}(\overline{w}^{mh})\Big]\ d\overline{y}'\Big) e_{mh\overline{x}'}(\overline{u}'^0) = \\ &= a^H_{ijmh} e_{mh\overline{x}'}(\overline{u}'^0), \quad \overline{x}' \in \Omega',\ i, j = 1, 2, 3, \end{aligned} \tag{18}$$

where

$$a^H_{ijmh} = \frac{1}{|Y'|}\int_{Y'_S} a^\epsilon_{ijkl}\Big[\delta_{km}\delta_{lh} + e_{kl\overline{y}'}(\overline{w}^{mh})\Big]\ d\overline{y}', \tag{19}$$

$(i, j, m, h = 1, 2, 3)$ represent the homogenized elasticity tensor.

Integrating the equation (14) on Y'_S and using the $Y'-$ periodically condition of the functions $\overline{u}'^1$ and $\overline{u}'^2$, we obtain the homogenized equations (macroscopic equations):

$$\frac{\partial}{\partial x_j}\Big[(1 - \Pi_Y) a_{ijmh} e_{mh\overline{x}}(\overline{u}^0) + \Big(\frac{1}{|Y'|}\int_{Y'_S} a_{ijkl} e_{kl\overline{y}'}(\overline{w}^{mh})\ d\overline{y}'\Big) e_{mh\overline{x}}(\overline{u}^0)\Big] = -\widetilde{f}_i, \tag{20}$$

în Ω, $i = 1, 2, 3$, in the dimensional form, for that $\frac{|Y'_S|}{|Y'|} = \frac{|Y'| - |Y'_V|}{|Y'|} = 1 - \frac{|Y'_V|}{|Y'|} = 1 - \Pi_Y$, where $\widetilde{f}'_i = \dfrac{1}{|Y'|}\displaystyle\int_{Y'_S} \rho_0 f'_i\ d\overline{y}'$. These equations describes the macroscopic laws.

2. a^H_{ijkl}, $\overline{w}^{mh}$, σ^H_{ij} - properties

Following the ideas from [4], we are interested in establishing some properties of the homogenized tensor in order to characterize σ^H_{ij}. These properties are then used for the homogenized equations study.

Remark 1. The following properties are valid:

a)
$$a^H_{ijkl} = (1-\Pi_Y)a_{ijkl}, \quad \forall i,j,k,l \in \{1,2,3\}, \text{ with } \ i \neq j. \tag{21}$$

b)
$$\int_{Y'_S} a_{iimh} e_{mh\overline{y}'}(\overline{w}^{ii}) d\overline{y}' = \int_{Y'_S} a_{kkmh} e_{mh\overline{y}'}(\overline{w}^{kk}) d\overline{y}', \quad \forall i,k = 1,2,3, \tag{22}$$

$$\int_{Y'_S} a_{iimh} e_{mh\overline{y}'}(\overline{w}^{ii}) d\overline{y}' = \int_{Y'_S} a_{iimh} e_{mh\overline{y}'}(\overline{w}^{kk}) d\overline{y}', \quad \forall i,k = 1,2,3, \tag{23}$$

$$\int_{Y'_S} \nabla_{\overline{y}'} \cdot \overline{w}^{11} d\overline{y}' = \int_{Y'_S} \nabla_{\overline{y}'} \cdot \overline{w}^{22} d\overline{y}' = \int_{Y'_S} \nabla_{\overline{y}'} \cdot \overline{w}^{33} d\overline{y}'. \tag{24}$$

c)
$$\sigma^H_{ij}(\overline{x},t) = (1-\Pi_Y)a_{ijkl}e_{kl\overline{x}}(\overline{u}^0) + b\varphi(\overline{x},t)\delta_{ij}, \tag{25}$$

with $b = \lambda + \frac{2}{3}\mu$ (bulk modulus) and $\varphi(\overline{x},t) = \Big(\frac{1}{|Y'|}\int_{Y'_S} \nabla_{\overline{y}'} \cdot \overline{w} d\overline{y}'\Big)(\nabla_{\overline{x}} \cdot \overline{u}^0)$.

3. The microscopic problem solution's convergence

Some complementary results are presented in this section.

Remark 2. The characteristic function φ of Y'_S belongs to $L^2(Y')$ and its mean value has the form:

$$\widetilde{\varphi} = \frac{1}{|Y'|}\int_{Y'} \varphi \ d\overline{y}' = \frac{1}{|Y'|}\int_{Y'_S} \varphi \ d\overline{y}' = \frac{|Y'_S|}{|Y'|} = 1 - \Pi_Y.$$

Similarly, $\varphi^\epsilon(\overline{x}') = \varphi(\overline{y}')$, $\overline{x}' = \epsilon\overline{y}'$, the characteristic function of ω'^ϵ belongs to $L^2(\omega'^\epsilon_S)$. This function can be extended to $\varphi^\epsilon_e \in L^2(\Omega')$ of φ^ϵ weak convergent in $L^2(\Omega')$ toward $\widetilde{\varphi} = 1 - \Pi_Y$ (conform [2]).

Let us the Hilbert space $\mathcal{V} = (H^1(\Omega'^\epsilon))^3$ with the scalar product

$$(\overline{u},\overline{v})_{\mathcal{V}} = \sum_{i=1}^{3}(\nabla u_i, \nabla v_i)_{L^2(\Omega'^\epsilon)}, \ \overline{u} = (u_1,u_2,u_3), \overline{v} = (v_1,v_2,v_3) \in \mathcal{V} \tag{26}$$

and the norm

$$\|\overline{v}\|_{\mathcal{V}} = \Big(\sum_{i=1}^{3} \|\nabla v_i\|^2_{L^2(\Omega'^{\epsilon})}\Big)^{\frac{1}{2}},\ \overline{v} = (v_1, v_2, v_3) \in \mathcal{V}. \tag{27}$$

Let us assume that $\overline{f}' \in \mathcal{V}'$.

The microscopic problem for homogeneous boundary conditions is defined:

$$\begin{cases} \frac{1}{L}\frac{\partial}{\partial x_j}[\frac{1}{L}a^{\epsilon}_{ijkl}e_{kl}(\overline{u}'^{\epsilon})] = -\rho^{\epsilon}_0 {f'}_i(\overline{x}') \ \text{ in } \Omega'^{\epsilon} \\ \\ n_j\Big(\frac{1}{L}a^{\epsilon}_{ijkl}e_{kl}(\overline{u}'^{\epsilon})\Big) = 0 \ \text{ on } \partial\Omega'^{\epsilon} \setminus \Gamma'^{\epsilon} \end{cases} \quad (i = 1, 2, 3). \tag{28}$$

The variational formulation on $\mathcal{V}$, is corresponding by:

$$\begin{cases} \text{Find } \overline{u}'^{\epsilon} \in \mathcal{V} \text{ such that} \\ \dfrac{1}{L^2}\displaystyle\int_{\Omega'^{\epsilon}} a^{\epsilon}_{ijkl}e_{kl}(\overline{u}'^{\epsilon})e_{ij}(\overline{v})\ d\overline{x}' = \int_{\Omega'^{\epsilon}} \rho^{\epsilon}_0 {f'}_i v_i\ d\overline{x}' \ \ \forall \overline{v} \in \mathcal{V} \end{cases}, \tag{29}$$

which is equivalent to

$$\begin{cases} \text{Find } \overline{u}'^{\epsilon} \in \mathcal{V} \text{ such that} \\ a(\overline{u}'^{\epsilon}, \overline{v}) =< F, \overline{v} >,\ \forall \overline{v} \in \mathcal{V}, \end{cases} \tag{30}$$

where

$$a(\overline{u}, \overline{v}) = \frac{1}{L^2}\int_{\Omega'^{\epsilon}} a^{\epsilon}_{ijkl}e_{kl}(\overline{u})e_{ij}(\overline{v})\ d\overline{x}',\ \forall \overline{u},\ \overline{v} \in \mathcal{V}, \tag{31}$$

is a bilinear elliptic form on $\mathcal{V} \times \mathcal{V}$ [3], and

$$< F, \overline{v} >= \int_{\Omega'^{\epsilon}} \rho^{\epsilon}_0 {f'}_i v_i\ d\overline{x}', \tag{32}$$

a linear continuous functional on $\mathcal{V}$ with

$$\|F\|_{\mathcal{V}'} \leq \|\rho_0 \overline{f}'\|_{\mathcal{V}'} \leq \|\rho_0 \overline{f}'\|_{(L^2(\Omega'))^3}. \tag{33}$$

Theorem 1. Let $\overline{f}' \in \mathcal{V}'$. The problem (29) has a unique solution $\overline{u}'^{\epsilon} \in \mathcal{V}$. Moreover, there exist a positive constant C_1 such that

$$\|\overline{u}'^{\epsilon}\|_{\mathcal{V}} \leq C_1\|\rho_0 \overline{f}'\|_{(L^2(\Omega'))^3}. \tag{34}$$

Proof. The hypothesis of the Lax-Milgram theorem are satisfied, i.e. the functional F is continuous on $\mathcal{V}$, the bilinear form $a = a(\overline{u}, \overline{v})$ is continuous and $\mathcal{V}$ - eliptic, so the problem (29) has a unique solution $\overline{u}'^{\epsilon} \in \mathcal{V}$. The Korn's type inequality is valid:

$$\|\overline{u}'^{\epsilon}\|_{(H^1(\Omega'^{\epsilon}))^3} \leq C_K\Big[\|\overline{u}'^{\epsilon}\|_{(L^2(\Omega'^{\epsilon}))^3} + \Big(\int_{\Omega'^{\epsilon}} |e_{ij}(\overline{u}'^{\epsilon})|^2 \, d\overline{x}'\Big)^{1/2}\Big], \tag{35}$$

where $C_K = c_K(\Omega'^{\epsilon})$. The elasticity tensor (a^{ϵ}_{ijkl}) is positive definite one, so there exist a constant $C > 0$ such that

$$a^{\epsilon}_{ijkl} e_{kl}(\overline{u}'^{\epsilon}) e_{ij}(\overline{u}'^{\epsilon}) \geq C e_{ij}(\overline{u}'^{\epsilon}) e_{ij}(\overline{u}'^{\epsilon}).$$

We obtain:

$$\Big|a(\overline{u}'^{\epsilon}, \overline{u}'^{\epsilon})\Big| = \Big|\frac{1}{L^2}\int_{\Omega'^{\epsilon}} a^{\epsilon}_{ijkl} e_{kl}(\overline{u}'^{\epsilon}) e_{ij}(\overline{u}'^{\epsilon}) \, d\overline{x}'\Big| \geq C\Big|\frac{1}{L^2}\int_{\Omega'^{\epsilon}} e_{ij}(\overline{u}'^{\epsilon}) e_{ij}(\overline{u}'^{\epsilon}) \, d\overline{x}'\Big| \geq$$

$$\geq \tfrac{C}{L^2 \cdot C_K}\|\overline{u}'^{\epsilon}\|^2_{\mathcal{V}}, \tag{36}$$

and

$$| < F, \overline{u}'^{\epsilon} > | \leq \|F\|_{\mathcal{V}'} \cdot \|\overline{u}'^{\epsilon}\|_{\mathcal{V}} \leq \|\rho_0 \overline{f}'\|_{(L^2(\Omega'))^3} \cdot \|\overline{u}'^{\epsilon}\|_{\mathcal{V}}, \tag{37}$$

respectively. The inequalities (36), (37) yield

$$\|\overline{u}'^{\epsilon}\|^2_{\mathcal{V}} \leq \frac{L^2 C_K}{C} \cdot \Big|a(\overline{u}'^{\epsilon}, \overline{u}'^{\epsilon})\Big| = \frac{L^2 \cdot C_K}{C} \cdot | < F, \overline{u}'^{\epsilon} > | \leq \frac{L^2 \cdot C_K}{C} \cdot \|\rho_0 \overline{f}'\|_{(L^2(\Omega'))^3} \cdot \|\overline{u}'^{\epsilon}\|_{\mathcal{V}},$$

so

$$\|\overline{u}'^{\epsilon}\|_{\mathcal{V}} \leq C_1 \cdot \|\rho_0 \overline{f}'\|_{(L^2(\Omega'))^3}, \quad C_1 = \frac{L^2 C_K}{C}.$$

The proof is complete.

For $l, m \in \{1, 2, 3\}$ the functions $\overline{P}^{lm}(\overline{y}') = \Big(P^{lm}_k(\overline{y}')\Big)^3_{k=1}$ are defined

$$P^{lm}_k(\overline{y}') = y'_m \delta_{kl}, \ k = 1, 2, 3. \tag{38}$$

The solution $\overline{\chi}^{lm}(\overline{y}') = \Big(\chi^{lm}_k(\overline{y}')\Big)^3_{k=1}$ of the system

$$\begin{cases} \dfrac{\partial}{\partial y'_j}\Big(a^{\epsilon}_{ijkl}\dfrac{\partial(\chi^{lm}_k - P^{lm}_k)}{\partial y'_h}\Big) = 0, \quad \text{in } Y'_S, \ i = 1, 2, 3 \\ \chi^{lm}_k \ Y' - \text{periodics} \\ \frac{1}{|Y'|}\displaystyle\int_{Y'_S} \chi^{lm}_k \, d\overline{y}' = 0, \end{cases} \tag{39}$$

and $\overline{w}^{kh}_\epsilon(\overline{x}')$,

$$\overline{w}^{kh}_\epsilon(\overline{x}') = \overline{P}^{kh}(\overline{x}') - \epsilon\overline{\chi}^{kh}(\frac{\overline{x}'}{\epsilon}),\ \overline{x}' \in \Omega',\ k,h = 1,2,3, \tag{40}$$

are considered as in Ciorănescu D. [3], also

$$(\eta^{kl}_\epsilon(\overline{x}'))_{ij} = a^\epsilon_{ijmh}(\overline{x}')e_{mh\overline{y}'}(\overline{w}^{kl})\ [?]. \tag{41}$$

Theorem 2. Let $\overline{f}' \in \mathcal{V}'$. Then, there exists an extension $\overline{u}'^\epsilon_p$ of the solution $\overline{u}'^\epsilon$ of the problem (29), from Ω'^ϵ to Ω' and a sequence $\epsilon_n \to 0$ such that:

$$\begin{cases} \text{i. } \overline{u}'^{\epsilon_n}_p \rightharpoonup \overline{u}'^0 \text{ in } (H^1(\Omega'))^3, \\ \text{ii. } \overline{u}'^{\epsilon_n}_p \to \overline{u}'^0 \text{ in } (L^2(\Omega'))^3, \\ \text{iii. } a^{\epsilon_n}_{ijkl}e_{kl}(\overline{u}'^{\epsilon_n}_p) \rightharpoonup a^H_{ijkl}e_{kl\overline{x}'}(\overline{u}'^0) \text{ in } (L^2(\Omega'))^{3\times 3}, \end{cases}$$

where $\overline{u}'^0 = (u'^0_1, u'^0_2, u'^0_3)$ is the unique solution in $(H^1(\Omega'))^3$ of the homogenized problem

$$\begin{cases} \dfrac{1}{L^2}\dfrac{\partial}{\partial x'_j}\Big(a^H_{ijkl}e_{kl\overline{x}'}(\overline{u}'^0)\Big) = -\widetilde{f}'_i,\ \text{in } \Omega' \\ \\ \frac{1}{L}a^H_{ijkl}e_{kl\overline{x}'}(\overline{u}'^0) = 0,\ \text{on } \partial\Omega' \end{cases},\quad i = 1,2,3, \tag{42}$$

and (a^H_{ijkl}) defined by (19).

Proof. From **Theorem 1**, we have

$$\|\overline{u}'^\epsilon\|_{(H^1(\Omega'^\epsilon))^3} \le C_1\|\rho_0\overline{f}'\|_{(L^2(\Omega'))^3}.$$

For $\sigma^\epsilon = (\sigma^\epsilon_{ij})^3_{i,j=1}$ defined by $\sigma^\epsilon_{ij} = a^\epsilon_{ijkl}e_{kl}(\overline{u}'^\epsilon)$ the following inequality stands [3]:

$$\|\sigma^\epsilon\|_{(L^2(\Omega'^\epsilon))^3} \le C.$$

All these estimations [2], **Lemma 3**, lead the conclusion: there exists an extension $\overline{u}'^\epsilon_p$ of $\overline{u}'^\epsilon$ of (29), from Ω'^ϵ to Ω' and a sequence $\epsilon_n \to 0$ such that:

- $\|\overline{u}'^\epsilon_p\|^2_{H^1(\Omega')} \le c^2\|\overline{u}'^\epsilon_p\|^2_{H^1(\Omega'^\epsilon)} \le c^2C_1^2\|\rho_0\overline{f}'\|^2_{(L^2(\Omega'))^3}$,

$$\begin{cases} \text{i. } \overline{u}'^{\epsilon_n}_p \rightharpoonup \overline{u}'^0,\ \text{in } (H^1(\Omega'))^3, \\ \\ \text{ii. } \overline{u}'^{\epsilon_n}_p \to \overline{u}'^0,\ \text{in } (L^2(\Omega'))^3, \\ \\ \text{iii. } \sigma^{\epsilon_n}_p \rightharpoonup \widetilde{\sigma}^0,\ \text{in } (L^2(\Omega'))^{3\times 3}, \end{cases} \tag{43}$$

respectively, where $\sigma_p^{\epsilon_n} = (\sigma_{ij\ p}^{\epsilon_n})$, with $\sigma_{ij\ p}^{\epsilon_n} = a_{ijkl}^{\epsilon_n} e_{kl}(\overline{u}'^{\epsilon_n}_p)$.
The variational formulation of (28), point out that the tensor $(\sigma_{ij}^{\epsilon_n})$ verifies the equation:

$$\frac{1}{L}\int_{\Omega'^{\epsilon_n}} \sigma_{ij}^{\epsilon_n} e_{ij\overline{x}'}(\overline{v})\ d\overline{x}' =< \rho_0 \overline{f}', \overline{v} >_{\mathcal{V}',\mathcal{V}},\ \forall \overline{v} \in \mathcal{V}. \tag{44}$$

Let us evaluate all the terms in (44):

$$\frac{1}{L}\int_{\Omega'^{\epsilon_n}} \sigma_{ij}^{\epsilon_n} e_{ij\overline{x}'}(\overline{v})\ d\overline{x}' = \frac{1}{L}\int_{\Omega'} \varphi_e^{\epsilon_n} \sigma_{ij}^{\epsilon_n} e_{ij\overline{x}'}(\overline{v})\ d\overline{x}' \to \frac{1}{L}\int_{\Omega'} \widetilde{\sigma}_{ij}^0 e_{ij\overline{x}'}(\overline{v})\ d\overline{x}' \tag{45}$$

for $\epsilon_n \to 0$. The right side of the equation (44) can be written for $\epsilon_n \to 0$:

$$< \rho_0 \overline{f}', \overline{v} >_{\mathcal{V}',\mathcal{V}} = \int_{\Omega'^{\epsilon_n}} \rho_0 f_i' v_i\ d\overline{x}' = \int_{\Omega'} \varphi_e^{\epsilon_n} \rho_0 f_i' v_i\ d\overline{x}' \to < \widetilde{f}', \overline{v} >, \tag{46}$$

Taking into account $(43)_3$, we get:

$$\frac{1}{L}\int_{\Omega'} \widetilde{\sigma}_{ij}^0 e_{ij\overline{x}'}(\overline{v})\ d\overline{x}' =< \widetilde{f}', \overline{v} >,\ \forall \overline{v} \in (\mathcal{C}_0^\infty(\Omega'))^3. \tag{47}$$

The proof is completed if:

$$\widetilde{\sigma}_{ij}^0 = a_{ijkl}^H e_{kl\overline{x}'}(\overline{u}'^0). \tag{48}$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega')$, $\phi \overline{w}_{\epsilon_n}^{kl}$ test function in (44) and $\overline{v} = \phi \overline{u}'^{\epsilon_n}$ in relation

$$\int_{\Omega'} \eta_\epsilon^{kl} e_{kl\overline{x}'}(\overline{v})\ d\overline{x}' = 0, \quad \forall \overline{v} \in (H_0^1(\Omega'))^3.$$

We have:

$$\frac{1}{L}\int_{\Omega'^{\epsilon_n}} a_{ijmh}^{\epsilon_n} e_{mh}(\overline{u}'^{\epsilon_n}) e_{ij}(\overline{w}_{\epsilon_n}^{kl})\phi\ d\overline{x}' + \frac{1}{2L}\int_{\Omega'^{\epsilon_n}} \sigma_{ij}^{\epsilon_n}\Big[(\overline{w}_{\epsilon_n}^{kl})_i \frac{\partial \phi}{\partial x_j'} + (\overline{w}_{\epsilon_n}^{kl})_j \frac{\partial \phi}{\partial x_i'}\Big]\ d\overline{x}'$$
$$=< \rho_0 \overline{f}', \phi \overline{w}_{\epsilon_n}^{kl} > \int_{\Omega'^{\epsilon_n}} (\eta_{\epsilon_n}^{kl})_{ij} e_{ij}(\overline{u}'^{\epsilon_n})\phi\ d\overline{x}' + \frac{1}{2}\int_{\Omega'^{\epsilon_n}} (\eta_{\epsilon_n}^{kl})_{ij}\Big[u'^{\epsilon_n}_i \frac{\partial \phi}{\partial x_j'} + u'^{\epsilon_n}_j \frac{\partial \phi}{\partial x_i'}\Big]\ d\overline{x}' = 0. \tag{49}$$

The symmetry of $(a_{ijkl}^{\epsilon_n})$, that assumes

$$\sigma_{ij}^{\epsilon_n} e_{ij}(\overline{w}_{\epsilon_n}^{kl}) = (\eta_{\epsilon_n}^{kl})_{ij} e_{ij}(\overline{u}'^{\epsilon_n}).$$

The equation (49) are equivalent to

$$\begin{cases} \frac{1}{2L}\int_{\Omega'^{\epsilon_n}} \sigma_{ij}^{\epsilon_n}\Big[(\overline{w}_{\epsilon_n}^{kl})_i\frac{\partial\phi}{\partial x'_j}+(\overline{w}_{\epsilon_n}^{kl})_j\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}'- \\ -\frac{1}{2L}\int_{\Omega'^{\epsilon_n}}(\eta_{\epsilon_n}^{kl})_{ij}\Big[u'^{\epsilon_n}_i\frac{\partial\phi}{\partial x'_j}+u'^{\epsilon_n}_j\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}' =< \rho_0\overline{f}', \phi\overline{w}_{\epsilon_n}^{kl} >, \end{cases} \tag{50}$$

i.e.,

$$\begin{cases} \frac{1}{2L}\int_{\Omega'} \varphi_e^{\epsilon_n}\sigma_{ij}^{\epsilon_n}\Big[(\overline{w}_{\epsilon_n}^{kl})_i\frac{\partial\phi}{\partial x'_j}+(\overline{w}_{\epsilon_n}^{kl})_j\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}'- \\ -\frac{1}{2L}\int_{\Omega'}\varphi_e^{\epsilon_n}(\eta_{\epsilon_n}^{kl})_{ij}\Big[u'^{\epsilon_n}_i\frac{\partial\phi}{\partial x'_j}+u'^{\epsilon_n}_j\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}' = \int_{\Omega'}\varphi_e^{\epsilon_n}\rho_0 f'_i\phi(\overline{w}_{\epsilon_n}^{kl})_i\ d\overline{x}'. \end{cases} \tag{51}$$

For $\epsilon_n \to 0$, in (51), becomes:

$$\begin{cases} \frac{1}{2L}\int_{\Omega'} \widetilde{\sigma}_{ij}^{0}\Big[y'_l\delta_{ki}\frac{\partial\phi}{\partial x'_j}+y'_l\delta_{kj}\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}'- \\ -\frac{1}{2L}\int_{\Omega'}a_{ijkl}^{H}\Big[u'^{0}_i\frac{\partial\phi}{\partial x'_j}+u'^{0}_j\frac{\partial\phi}{\partial x'_i}\Big]\ d\overline{x}' = \Big\langle \widetilde{f}', \phi\overline{P}^{kl}\Big\rangle. \end{cases} \tag{52}$$

which can be rewritten as:

$$\frac{1}{L}\int_{\Omega'}\widetilde{\sigma}_{ij}^{0}e_{ij}(\phi\overline{P}^{kl})\ d\overline{x}' - \frac{1}{L}\int_{\Omega'}\widetilde{\sigma}_{kl}^{0}\phi\ d\overline{x}' + \frac{1}{2L}\int_{\Omega'}a_{klij}^{H}e_{ij}(\overline{u}'^{0})\phi\ d\overline{x}' = \Big\langle \widetilde{f}', \phi\overline{P}^{kl}\Big\rangle. \tag{53}$$

If we write (47), for the test function $\phi\overline{P}^{kl}$, the main result of the theorem is obtained

$$\int_{\Omega'}\widetilde{\sigma}_{kl}^{0}\phi\ d\overline{x}' = \int_{\Omega'}a_{klij}^{H}e_{ij}(\overline{u}'^{0})\phi\ d\overline{x}'\ \ \phi\in\mathcal{C}_0^{\infty}(\Omega'), \tag{54}$$

and

$$\widetilde{\sigma}_{kl}^{0} = a_{klij}^{H}e_{ij}(\overline{u}'^{0}).$$

The proof is complete.

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