A NOTE ON EXISTENCE AND UNIQUENESS OF THE PERTURBED EVOLUTION FAMILY IN BANACH SPACES

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Abstract. The aim of this paper is to study the existence of a unique solution to the linear Volterra integral equation

$$U_B\,(t,s) = U\,(t,s) + \int_s^t U\,(t,\tau)\,B\,(\tau)\,U_B\,(\tau,s)\;d\tau,\;\; t,s\in\mathbb{R}$$

without assuming the exponentially bounded condition for the reversible evolution family $\mathcal{U}=\{U(t,s)\}_{t,s\in\mathbb{R}}$ and to prove that this solution is a reversible evolution family, too.

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1. Preliminaries

Let X be a real or complex Banach space and let $\Delta=\{(t,s)\in\mathbb{R}^2 : t\geq s\}$. The norm on X and on $\mathcal{B}(X)$ – the Banach algebra of bounded linear operators on X, will be denoted by $\|\cdot\|$. We now recall the definition of evolution family:

Definition 1. A family of bounded linear operators $\mathcal{U}=\{U(t,s)\}_{(t,s)\in\Delta}$ on X is an *evolution family* if

(e_1) $U(t,t)=I$ (the identity operator on X) for $t\in\mathbb{R}$,

(e_2) $U(t,s)U(s,t_0)=U(t,t_0)$, for all $t\geq s\geq t_0$.

If, in addition, there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\| U(t,s) \| \leq M e^{\omega(t-s)} \text{ for all } t \geq s,$$

the evolution family $\mathcal{U}$ is said to be *exponentially bounded*. The evolution family $\{U(t,s)\}_{t\geq s}$ is *strongly continuous* if the function $\Delta \ni (t,s) \longmapsto U(t,s)x \in X$ is continuous for every $x \in X$.

The notion of evolution family arises naturally from the theory of well-possed non-autonomous Cauchy problems. In fact, an evolution family arises from the following well-posed evolution equation

$$\dot{u}(t) = A(t)u(t),\ t \geq s,$$

where $A(t) : D(A(t)) \subset X \longrightarrow X$ are (in general, unbounded) linear operators, for $t \geq s$. Roughly speaking, when the Cauchy problem

$$\begin{cases} \dot{u}(t) = A(t)u(t),\ t \geq s \\ u(s) = x \end{cases} \tag{1}$$

is well-posed with regularity subspaces $(Y_t)_{t\in\mathbb{R}}$, then the operator

$$U(t,s)x := u(t;s,x) \text{ for } t \geq s \text{ and } x \in Y_s,$$

where $u(\cdot;s,x)$ is the unique solution of (1), can be extended by continuity to a bounded linear operator on X. Moreover, the family $\{U(t,s)\}_{t\geq s}$ is a strongly continuous evolution family on X. For more details on well-posed non-autonomous Cauchy problems we refer the reader to Nagel and Nickel [5] and the references therein.

A classical example of evolution families is given by:

Example 2. *Set $u : \mathbb{R} \longrightarrow (0,\infty)$. The operator*

$$U(t,s)x := \frac{u(t)}{u(s)}x, \text{ for } t \geq s \text{ and } x \in X$$

generates an evolution family on a real Banach space X. Moreover, if the function $u(\cdot)$ is continuous, then $\{U(t,s)\}_{t\geq s}$ is a strongly continuous evolution family.

It is easy to see that the operators $U(t,s)$ from above are invertible for all $t \geq s$. Moreover, by setting $U(s,t) := U(t,s)^{-1}$ for $t > s$, the evolution family $\mathcal{U}$ can be extended to a family $\{U(t,s)\}_{(t,s)\in\mathbb{R}^2}$. In this case, relation (e_2) holds for all $t, s, t_0 \in \mathbb{R}$.

Definition 3. A family of bounded linear operators $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}^2}$ on X is a *reversible evolution family* if

$$U(t,t) = I \text{ and } U(t,s)U(s,t_0) = U(t,t_0) \text{ for } t, s, t_0 \in \mathbb{R}.$$

If, in addition, the map $\mathbb{R}^2 \ni (t,s) \longmapsto U(t,s)x \in X$ is continuous for every $x \in X$, the evolution family is *strongly continuous*. We say that a reversible evolution family is *exponentially bounded* if there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\| U(t,s) \| \leq Me^{|t-s|}, \text{ for all } t, s \in \mathbb{R}.$$

Remark 4. *If $\{U(t,s)\}_{t,s\in\mathbb{R}^2}$ is a reversible evolution family then the operator $U(t,s)$ is invertible and $U(t,s)^{-1} = U(s,t)$.*

In the last years, some important results concerning the existence of a unique solution to the linear Volterra integral equation

$$U_B(t,s) = U(t,s) + \int_s^t U(t,\tau)\, B(\tau)\, U_B(\tau,s)\, d\tau \tag{2}$$

were obtained. In the following we review some of them.

Schnaubelt showed in [2, 9.19 pp. 487] a perturbation result for exponentially bounded evolution families:

Theorem 5. *Let $\{U(t,s)\}_{(t,s)\in\Delta}$ be a strongly continuous exponentially bounded evolution family and $B(t) : D(B(t)) \subset X \longrightarrow X$, $t \in \mathbb{R}$ be closed operators on the Banach space X such that*

(i) *$U(t,s)X \subset D(B(t))$, for $t \geq s$;*

(ii) *the function $t \longmapsto B(t)U(t,s)$ is strongly continuous;*

(iii) *there is a locally integrable function $k : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with*

$$\| B(t)U(t,s) \| \leq k(t-s), \text{ for } t > s.$$

Then there is a unique strongly continuous exponentially bounded evolution family $\{U_B(t,s)\}_{t\geq s}$ *such that*

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\, d\tau,$$

for $t \geq s$ *and* $x \in X$*. Moreover, for* $(s,x) \in \mathbb{R}\times X$ *we have* $U_B(t,s)x \in D(B(t))$ *for almost all* $t > s$*, the function* $[s,\infty) \ni t \longmapsto B(t)U_B(t,s)x \in X$ *is locally integrable and*

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\, d\tau,$$

for all $t \geq s$ *and* $x \in X$*.*

Notice that if the family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is generated by (1) then the function $t \longmapsto U(t,s)x$ can be consider as a mild solution of the non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = [A(t)+B(t)]u(t) \\ u(s) = x \end{cases}.$$

Theorem 5 implies that if $B : \mathbb{R} \longrightarrow \mathcal{B}(X)$ is a strongly continuous and bounded operator-valued function then the integral equation (2) has a unique solution. Moreover, this solution generates a strongly continuous exponentially bounded evolution family.

To prove these results author used the evolution semigroup associated with the exponentially bounded evolution family $\{U(t,s)\}_{t\geq s}$. Unfortunately, most of the (reversible) evolution families failed to be exponentially bounded (for instance the evolution family considered in Example 2 for $u(t) = e^{e^t}$, is not exponentially bounded on $X = \mathbb{R}$).

Recently, L.H. Popescu [6] proposed a new approach based on the following result obtained by B. Rzepecki in [7]:

Lemma 6 (Theorem 1 in [7]). *Let* $K : \Delta \to \mathcal{B}(X)$ *be a strongly continuous operator-valued function such that*

$$\| K(t,s) \| \leq e^{\int_s^t \omega(\tau)d\tau}, \text{ for all } t \geq s,$$

for some locally integrable function $\omega : \mathbb{R} \longrightarrow (0,\infty)$*, and let* $B : \mathbb{R} \longrightarrow \mathcal{B}(X)$ *be a strongly continuous and bounded operator-valued function. Then for any*

$x \in E$ and $s \in \mathbb{R}$, the integral equation

$$y(t) = K(t,s)x + \int_s^t K(t,\tau)B(\tau)y(\tau)d\tau,\ t \geq s,$$

has a unique continuous solution, $y_{s,x} : [s,\infty) \longrightarrow X$. Moreover, for every $s \in \mathbb{R}$, the function

$$X \ni x \longmapsto y_{s,x}(\cdot) \in C([s,\infty), X)$$

is continuous in the topology of uniform convergence on compact subsets of $[s,\infty)$.

In fact, Popescu [6] showed that if $\{U(t,s)\}_{t\geq s}$ is a strongly continuous evolution family such that

$$\| U(t,s) \| \leq e^{\int_s^t \omega(\tau)d\tau},\ \text{for all } t \geq s,$$

for some locally integrable function $\omega : \mathbb{R} \longrightarrow (0,\infty)$, and $B : \mathbb{R} \longrightarrow \mathcal{B}(X)$ is a strongly continuous and bounded operator-valued function then the unique solution of equation (2) is an evolution family.

2. The main results

The aim of this paper is to extend the above result for reversible evolution families, showing that under certain conditions imposed to a strongly continuous reversible evolution family $\{U(t,s)\}_{t,s\in\mathbb{R}}$, the equation (2) has a unique solution and this solution is also a reversible evolution family.

Lemma 7. *Let $K : \{(t,s) \in \mathbb{R}^2 : t \leq s\} \to \mathcal{B}(X)$ be a strongly continuous operator-valued function such that*

$$\| K(t,s) \| \leq e^{\int_t^s \omega(\tau)d\tau},\ \text{for all } t \leq s.$$

For any $x \in X$ and $s \in \mathbb{R}$, the integral equation

$$z(t) = K(t,s)x - \int_t^s K(t,\tau)B(\tau)z(\tau)d\tau,\ t \leq s$$

has a unique continuous solution $z_{s,x} : (-\infty, s] \longrightarrow X$. Moreover, for every $s \in \mathbb{R}$, the function

$$X \ni x \longmapsto z_{s,x}(\cdot) \in C((-\infty,s], X)$$

is continuous.

Proof. It results as Theorem 1 from [7], considering the Fréchet space $F = C((-\infty, s], X)$, endowed with the family of semi-norms

$$p_n(y) = \sup_{n \leq t \leq s} \| y(t) \|, \text{ for } y \in F \text{ and } n \in \mathbb{Z}.$$

□

Theorem 8. *Let $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ be a strongly continuous reversible evolution family, $B : \mathbb{R} \longrightarrow \mathcal{B}(E)$ be a strongly continuous and bounded operator-valued function with $\sup_{t\in\mathbb{R}} \| B(t) \| = \delta$ and let $\omega : \mathbb{R} \longrightarrow (0,\infty)$ be a locally integrable function for which*

$$\| U(t,s) \| \leq e^{\left|\int_s^t \omega(\tau)d\tau\right|}, \text{ for all } t,s \in \mathbb{R}. \tag{3}$$

Then equation (2) has a unique solution and this solution generates a reversible evolution family.

Proof. Using both Lemma 6 and Lemma 7, we obtain that for each $x \in X$, the integral equation

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)xd\tau, \text{ for } t,s \in \mathbb{R},$$

has a unique solution, given by

$$U_B(t,s)x = \begin{cases} y_{s,x}(t), & t \geq s \\ z_{s,x}(t), & t < s \end{cases}.$$

Moreover, this solution is continuous with respect to t. We first show that $U_B(t,s)$ is a bounded linear operator on X. Indeed, we have

$$\alpha U_B(t,s)x + \beta U_B(t,s)y =$$
$$= U(t,s)(\alpha x + \beta y) + \int_s^t U(t,\tau)B(\tau)\left[\alpha U_B(\tau,s)x + \beta U_B(\tau,s)y\right]d\tau,$$

for all $\alpha, \beta \in \mathbb{K}$ ($\mathbb{K}$ denotes the set of real or complex numbers) and $x, y \in X$. Therefore $\alpha V(\cdot,s)x + \beta V(\cdot,s)y$ is the unique solution of the integral equation

$$y(t) = U(t,s)(\alpha x + \beta y) + \int_s^t U(t,\tau)B(\tau)y(\tau)d\tau, \text{ for } t \geq s.$$

This means that $U_B(t,s)$ is a linear operator on X. Now let $t,s\in\mathbb{R}$ with $t\geq s$. Suppose that $\| x_n - x \| \longrightarrow 0$ ($x_n, x\in X,\ n\in\mathbb{N}$). Using the continuity of the map $X\ni x\longmapsto y_{s,x}(\cdot)\in C([s,\infty),X)$, we have $\| y_{s,x_n}(\cdot)-y_{s,x}(\cdot)\| \longrightarrow 0$ uniformly on every compact subsets $K\subset[s,\infty)$. Considering $K=K_t:=\{t\}$ it follows $\| y_{s,x_n}(t)-y_{s,x}(t)\| \longrightarrow 0$. Hence $\| U_B(t,s)x_n - U_B(t,s)x\| \longrightarrow 0$. Thus $U_B(t,s)$ is a bounded linear operator on X for $t\geq s$. This result can be obtained similarly for $t<s$.

The technique below is similar to that in Lemma 9 from [6]. The reader may notice that we do not just repeat the arguments there, we merely explain where condition (3) is needed. Indeed, we can easily get

$$U_B(t,s)U_B(s,t_0)-U_B(t,t_0)=\int_s^t U(t,\tau)B(\tau)\left[U_B(\tau,s)U_B(s,t_0)-U_B(\tau,t_0)\right]d\tau,$$

for all $t,s,t_0\in\mathbb{R}$ (the integral is considered in the strong operator topology sense (see [3, Theorem 3.8.2. pp. 85])). If we set $s,t_0\in\mathbb{R}$, putting $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_+$ given by

$$\varphi(t)=e^{-\left|\int_s^t\omega(\tau)d\tau\right|}\,\| U_B(t,s)U_B(s,t_0)x-U_B(t,t_0)x\|,$$

then we have

$$e^{\left|\int_s^t\omega(\tau)d\tau\right|}\varphi(t)\leq\delta\left|\int_s^t e^{\left|\int_\tau^t\omega(u)du\right|}e^{\left|\int_s^\tau\omega(u)du\right|}\varphi(\tau)d\tau\right|,$$

and hence

$$\varphi(t)\leq\delta\left|\int_s^t\varphi(\tau)d\tau\right|,\ \text{for all } t\in\mathbb{R}.$$

Using Gronwall's lemma, we obtain that $\varphi(t)=0$, for all $t\in\mathbb{R}$, which implies that $U_B(t,s)U_B(s,t_0)=U_B(t,t_0)$, for all $t,s,t_0\in\mathbb{R}$. Therefore $\{U_B(t,s)\}_{t,s\in\mathbb{R}}$ is a reversible evolution family. Moreover, the operator $U_B(t,s)$ is strongly continuous with respect to t.

□

Notice that condition (3) is far less restrictive as exponentially bounded condition. In fact, all the evolution families generated by differential equations verify this kind of inequality (see for example [1, pp. 101]). A similar condition was considered by J.S. Muldowney in [4].

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