A SURVEY ON THE FUNCTIONAL EQUATIONS RELATED WITH NOMOGRAPHY

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Abstract. In the current paper we present some functional equations that characterize the functions which are nomographicaly represented by various types of nomograms (plane or in space).

We will approach the nomographical representation of the pseudosums of two and three variables. A mention will also be made of the functional equations that have as a solution the pseudosums.

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1. Introduction

It is well known that the functions and equations which admit the nomographic representations of certain types, are divided into certain classes.
These classes are analytical characterized in two ways: either by equations with partial derivatives or by functional equations. Although the conditions given in the form of partial derivatives equations are applied directly and relatively simply, the functional equations have the advantage of being more general and requiring only the monotony and continuity of functions.
The idea of characterizing certain functions by functional equations, appears in N.A.Kolmogorov [8]. He considered the problem in connections with mathematical statistics, following the axiomatical characterization of the functions named quasiarithmetical means

$$M_n(x_1, x_2, ..., x_n) = F^{-1}\left[\frac{F(x_1) + F(x_2) + ... + F(x_n)}{n}\right] \tag{1}$$

where F is a continuos and strictly monotone function. Independently M.Na-gumo arrived at the same results [10].
A connection was made between nomography and functional equations subsequent to the appearance of the papers of J. Aczél [2]-[3], W. Blaschke [7], F. Radó [6],[11],[12]. This was due to the necessity to characterize "more simply" the conditions of nomographical representation. Such conditions, expressed by functional equations, as well as the establishment of the equations of nomogram's elements (especially of the scales) offer an interesting and profitable connection between these two branches of mathematics.
J. Aczél [2] studied the functions (1) in the case $n = 2$ and succeeded to find that functional equation which characterizes them is that of the bisymmetry

$$f[f(u,x), f(y,v)] = f[f(u,y), f(x,v)]. \tag{2}$$

Aczél [3]-[4] found the solutions of functional equation (2) in the form

$$f(x,y) = H^{-1}\left[aH(x) + bH(y) + c\right] \tag{3}$$

where H is a continuous and strictly monotone function and $a, b, c \in \mathbb{R}$.
Following these results F. Radó studied the functional equations which characterize certain types of nomograms [11],[12]. He introduced pseudosums for the functions of two variables of the form

$$f(x,y) = H^{-1}[F(x) + G(y)].$$

The functions f are nomographically representable by the nomograms with alignment points with three parallel straight right line scales.
In this respect a new definition of the nomogram became necessary. It was formulated by F.Radó [11] who used the definitions of Blaschke-Bol [7].
Definition 1. *A family of (marked) curves is a topological image of a fascicle of parallel straight lines. A (marked) system of two (marked) curves forms a (marked) net of curves if each curve in the first family crosses the curve of the other family in not more than one point. Three families of (marked) curves make up a (marked) tissue if they, considered two by two, form (marked) curves net.*
Definition 2. *A plane marked tissue is a nomogram with marked curves.*
By noting the marks of the curves of the families which form the tissue by z_1, z_2, z_3 the value

$$z_3 = f(z_1, z_2) \tag{4}$$

corresponds to the pair (z_1, z_2).
From the above definitions it results that the function f is continuous and strictly monotone. Then we have the theorem [11].
Theorem 1. *The equation* (4) *can be nomographically represented by a nomogram with marked lines if and only if the function* $f : D'_1 \subset \mathbb{R}^2 \to \mathbb{R}$ *is continuous and strictly monotone* (*with respect to each variable*).
Let us consider the equation

$$G(z_1, z_2, z_3) = 0, \tag{5}$$

where $G : D \subset \mathbb{R}^3 \to \mathbb{R}$ is a continuous function that defines each of its variables as an implicit function of the two variables.

The problem (known as the anamorphosis) is of building up a nomogram with marked straight right lines for the equation (5) or (4) by arbitrarily choosing the net of curves z_1, z_2 as a straight line net.
Theorem 2. [11] *The necessary and sufficient condition so that the equation* (4) (*where* f *is continuous and strictly monotone*) *be nomographically represented by a nomogram with marked straight lines is that it could be written in the form*, (*i.e.Soreau equation*)

$$\begin{vmatrix} f_1(z_1) & g_1(z_1) & h_1(z_1) \\ f_2(z_2) & g_2(z_2) & h_2(z_2) \\ f_3(z_3) & g_3(z_3) & h_3(z_3) \end{vmatrix} = 0 \tag{6}$$

where $f_i(z_i), g_i(z_i), h_i(z_i), i = \overline{1,3}$ *are continuous functions.*
Taking into account that, by a correlation of the nomogram's plane and by a convenient choice of the projective system of coordinates, the marked family of curves is transformed into a marked scale, we notice that the nomogram with alignment points with three marked scales is the dual image of a nomogram with marked straight lines.
In the current paper we present some functional equations that characterize the functions which are nomographically represented by various types of nomograms (plane or in space).
We will approach the nomographical representation of the pseudosums of two and three variables. Mention will also be made of the functional equations that have as the solution the pseudosums.

2. The functions of two and more variables which are pseudosums

We follow the characterization the functions of two and three variables by means the functional equations. This returns to find those functional equations which have as solution just the studied functions.

a) Pseudosums with two terms

The following canonical forms are known for the equations with three variables of the third nomographical order and which can be nomographically represented by nomograms with alignment points of genus zero

$$F(z_1) + G(z_2) = H(z_3) \quad or \quad F(z_1)G(z_2)H(z_3) = 1, \tag{7}$$

where F, G, H are the real continuous and strictly monotone functions of one variable. Thus the equations (7) contain three functions of one variable, F, G, H that are continuous and strictly monotone. The inverse functions F^{-1}, G^{-1} and H^{-1} are also continuous strictly monotone. The equations (7) can still be written under the form

$$z_3 = f(z_1, z_2) = H^{-1}[F(z_1) + G(z_2)] \quad or \quad z_3 = f(z_1, z_2) = H^{-1}[F(z_1).G(z_2)] \tag{8}$$

The function of two variables, f, defined by the equation (8), is named a *pseudosum with two terms* (it was first used by F.Radó [11]).

Definition 3. *The function of two variables $z_3 = f(z_1, z_2)$, defined in a plane domain is named pseudosum if the relation (8) are fulfilled in each point of the domain and the functions of one variable F, G, H are continuous and strictly monotone.*

Theorem 3. *The necessary and sufficient condition so that the continuous and strictly monotone function of two variables, $f(z_1, z_2)$, be the pseudosum of two terms is there exist the decompositions (8).*

F.Radó [11] has demonstrated that the pseudosums are continuous and strictly monotone solutions for the functional equation

$$f[\overline{f}(u, z_1), \widetilde{f}(z_2, v)] = f[\overline{f}(u, z_2), \widetilde{f}(z_1, v)]. \tag{9}$$

Here the functions $\overline{f}, \widetilde{f}$ are also defined by the equation (4) solved with respect to z_1, respectively z_2 (i.e. $z_1 = \overline{f}(z_2, z_3)$ and $z_2 = \widetilde{f}(z_3, z_1)$).

The functional equation (9),which characterizes the pseudosums, generalizes the bisymmetry equation (2).

b) Pseudosums with three terms

The functional equation with four unknown functions has been studied.

$$g[\varphi(z_1, z_2), z_3] = h[z_1, \psi(z_2, z_3)] \tag{10}$$

The equation (10) represents a generalization of the associativity equation. Therefore, we want to represent the functions of three variables $f(z_1, z_2, z_3)$ by superpositions of functions of one variable. So, we have

Theorem 4. [11] *The general, continuous and strictly monotone solutions of the equation* (10) *are the functions*

$$\begin{array}{rclrcl} \varphi(z_1,z_2) & = & H_1^{-1}[F_1(z_1)+G_1(z_2)], & g(z_1,z_2) & = & H_3^{-1}[H_1(z_1)+G_2(z_2)] \\ \psi(z_1,z_2) & = & H_2^{-1}[G_1(z_1)+G_2(z_2)], & h(z_1,z_2) & = & H_3^{-1}[F_1(z_1)+H_2(z_2)] \end{array} \tag{11}$$

where $F_1, G_1, G_2, H_1, H_2,$ *are arbitrary but continuous and strictly monotone functions.*

By replacing (11) in (10) we have

$$z_4 = f(z_1, z_2, z_3) = H_3^{-1}[F_1(z_1) + G_1(z_2) + G_2(z_3)] \tag{12}$$

With a convenient notation we obtain

$$z_4 = f(z_1, z_2, z_3) = K^{-1}[F(z_1) + G(z_2) + H(z_3)] \tag{13}$$

By an analogous way we obtain next preudosum

$$z_4 = f(z_1, z_2, z_3) = K_1^{-1}[F^*(z_1).G^*(z_2).H^*(z_3)]. \tag{13a}$$

Definition 4. *The function of three variables* (13) *or* (13*a*) *is a pseudosum of three terms if in every point of a domain from the* $\mathbb{R}^3$ *space there exist the relations* (13) *or* (13*a*) *while the functions* F, F^*, G, G^*, H, H^*, K, K_1 *are continuous and strictly monotone*

Therefore, we can enunciate the following

Theorem 5. [11] *The necessary and sufficient condition so that the continuous and strictly monotone functions of three variables be pseudosums with three terms is existence of the decomposition*

$$z_4 = f(z_1, z_2, z_3) = g[\varphi(z_1, z_2), z_3] = h[z_1, \psi(z_2, z_3)]. \tag{14}$$

3. Main result

Nomographical representations of the pseudosums

We will analyse the nomographical representations of the functions of two and more variables. These functions, named pseudosums, are solutions of the functional equations which have mentioned above.

i) The nomographical representation of the pseudosums with two terms

The equations with three variables of the third nomographic order, (7) are represented by nomograms with alignment points of genus zero, with three straight line scales. For these equations there are the Soreau forms, with d as a real parameter:

$$\begin{vmatrix} 0 & F(z_1) & 1 \\ d & G(z_2) & 1 \\ \dfrac{d}{2} & \dfrac{H(z_3)}{2} & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} d & \dfrac{1}{F(z_1)} & 1 \\ 0 & G(z_2) & 1 \\ \dfrac{d}{1-H(z_3)} & 0 & 1 \end{vmatrix} = 0. \tag{15}$$

These scales are situated on the three parallel straight lines and, respectively, on two parallel straight lines and the third that intersects them.

If the scales of the nomograms (for (7)) are situated either on three straight lines which cross each other in a point, or on three straight lines that form a triangle , then we have the Soreau equations

$$\begin{vmatrix} \dfrac{1}{F(z_1)} & 0 & 1 \\ 0 & \dfrac{1}{G(z_2)} & 1 \\ \dfrac{1}{H(z_3)} & \dfrac{1}{H(z_3)} & 1 \end{vmatrix} = 0, \begin{vmatrix} 0 & \dfrac{F(z_1)}{1+F(z_1)} & 1 \\ \dfrac{1}{1+G(z_2)} & 0 & 1 \\ \dfrac{H(z_3)}{H(z_3)-1} & \dfrac{1}{1-H(z_3)} & 1 \end{vmatrix} = 0. \tag{16}$$

Then, by adding the parameters that are necessary in the nomogram building up we can infer the equations for the nomogram scales.

So can establish a correspondence between the pseudosums and the nomograms with alignment points of genus zero.

Theorem 6. *The equation* (7), *i.e. the pseudosum of two variables* (8) *are nomographically represented by nomograms with alignment points of genus zero. Reciprocally, any nomogram of genus zero corresponds to the equation* (7), *or to a pseudosum* (8).

Remark 1. Taking into consideration the Theorem 3 and the point $3i$, we establish that the pseudosums (3) (a particular case of (8)) will be nomographi-

cally represented by nomograms with alignment points of genus zero with three homotetical straight line scales.
For $a = b = 1/2$ and $c = 0$ the pseudosums (3), analyzed by Aczél [2] will be nomographically represented by nomograms with alignment points, that have the three scales situated on three parallel equidistant straight lines (the scale z_3 is situated in the middle of the distance between the other two scales).
ii) Nomographical representations of pseudosums with three terms
The notions introduced in Section 1 referring to the representation of the equation with three variables (5,) and respectively of a function of two variables by a nomogram with marked lines; can be further employed in the case of an equation of four variables as well as of a function of three variables. This extension is performed on the purpose of representing them either by a nomogram in space with coplanarity points, or by a nomogram in space with families of surfaces. We will use the extended concepts of family of curves, net and tissues to a three-dimensional space [11], in order to represent the pseudosums functions by a space nomogram.
Definition 5. [11] *A family of surfaces from a domain $E \subset \mathbb{R}^3$ is the topological image of a system of parallel planes in the domain $E_1 \subset \mathbb{R}^3$. The real constants in the equations of the families of surfaces represent the marks of the families.*
Definition 6.*A system made up of three families of (marked) surfaces in $E \subset \mathbb{R}^3$ is called a (marked) net of surfaces in $E \subset \mathbb{R}^3$, if the three above families together with the domain E are homomorphicaly with the three families of parallel planes from $E_1 \subset \mathbb{R}^3$ (two by two orthogonal). A system of four families of (marked) surfaces is named a (marked) tissue in space if any group of three families from this system forms a (marked) net of surfaces.*
Definition 7. *A market tissue in space is named nomogram in space with marked surfaces.*

Let us consider a tissue in space consisting of three families of planes which are parallel with the coordinate planes and a forth family made of the level surfaces of the function (13), i.e.

$$z_4 = f(z_1, z_2, z_3) = K^{-1}[F(z_1) + G(z_2) + H(z_3)] = const.$$

By applying a topological transformation to this tissue in space so that the families of parallel planes remain parallel with the coordinate planes

$$z_1' = F(z_1), z_2' = G(z_2), z_3' = H(z_3),$$

we have $K^{-1}(z_1' + z_2' + z_3') = const.$ or $z_1' + z_2' + z_3' = const.$and we also obtain a family of parallel planes.

Such a tissue is named a *regular tissue in space*.

Let us build up different types of nomograms for the functions f of three variables:

$$z_4 = f(z_1, z_2, z_3) \tag{17}$$

such as compound plane nomograms with alignment points, nomograms in space with coplanar points and a regular tissue in space.

α) Subsequently if the function of three variables (or the equation of four variables),(17), admits a decomposition of the forms (10), namely

$$z_4 = g[\varphi(z_1, z_2), z_3] \quad or \quad z_4 = h[z_1, \psi(z_2, z_{3)}]$$

(i.e. the Goursat condition is satisfied), then we can build up a compound nomogram for them.

By noting

$$\xi = \varphi(z_1, z_2), \qquad z_4 = g[\xi, z_3] \tag{18}$$

the function f (or the equation (17)) can be represented by a compound plane nomogram consisting of two nomograms with marked lines. Each of these two nomograms contains two families of marked lines as well as an unmarked family (for variable ξ), common to both nomograms. If for each of the equations (18) we can write the Soreau equations, then the function f and the equation (17) can be nomographically represented by a compound plane nomogram with alignment points.

Since the pseudosums (13) are particular cases of both (17) and (18) then they can be represented by such nomograms and moreover all scales are situated on the parallel straight right lines. The relations (18) become

$$\xi = F(z_1) + G(z_2), \quad K(z_4) = \xi + H(z_3) \tag{19}$$

and the corresponding Soreau equations are

$$\begin{vmatrix} 0 & \xi & 1 \\ 2 & -F(z_1) & 1 \\ 1 & \frac{G(z_2)}{2} & 1 \end{vmatrix} = 0, \begin{vmatrix} 0 & \xi & 1 \\ 2 & H(z_3) & 1 \\ 1 & \frac{K(z_4)}{2} & 1 \end{vmatrix} = 0. \tag{20}$$

According to the Theorem 6, we can enunciate

Theorem 7. *The pseudosums with three terms* (13) *can be nomogra* $-$

phically represented by a compound plane nomogram with alignment points of genus zero and all parallel scales.
There exists and reciprocal theorem: *any compound nomograms with alignment points nomogram represents a pseudosum* (13).
β) If the functions (13) and (13a) are written as

$$F(z_1)+G(z_2)+H(z_3)+K(z_4)=0,\ \text{or}\ F^*(z_1)G^*(z_2)H^*(z_3)K_1(z_4)=1 \quad (21)$$

we obtain the functional equations with four unknown function of one variable. The equations (21) can be nomographically represented by nomograms in space with coplanar points with all four scales situated at straight right lines.

For the equations (21) the Soreau equations are

$$\begin{vmatrix} 0 & 0 & F(z_1) & 1 \\ 1 & 0 & G(z_2) & 1 \\ 0 & 1 & H(z_3) & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{K(z_4)}{3} & 1 \end{vmatrix}=0,\ \begin{vmatrix} F^*(z_1) & 0 & 0 & 1 \\ -\frac{1}{G^*(z_2)} & 1 & 0 & 1 \\ 0 & 1 & -H^*(z_3) & 1 \\ 0 & 0 & \frac{1}{K_1(z_4)} & 1 \end{vmatrix}=0. \quad (22)$$

The nomogram for the first equation (21) has all scales situated on the straight right lines parallel with $0Z$ axis, while that for the second equation has two parallel scales situated in $X0Y$ plane and two in the $Y0Z$ plane.

From (22) we can easy infer the parametric equations of the scales of the nomograms. In fact, we can also introduce the constants which represent the moduli of scales as well as the dimensions of the nomogram.

Therefore, we can enunciate
Theorem 8. *Pseudosums with three terms* (13) *can be nomographically represented by the nomograms in space with coplanare points.*
The reciprocal theorem also really exists.
γ) For the function (17) we can also build up a marked tissue in space but this will not be easy to illustrate while the corresponding nomogram (tissue) is difficult to use. An analogous theorem to those (7) and (8) can also be inferred in the case γ.

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