ESTIMATION OF THE NUMBER OF CRITICAL POINTS OF CIRCLE - VALUED MAPPINGS

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Abstract. Circle - valued Morse functions deals with functions of the form $f : M \to S^1$ having only nondegenerate critical points [5], [8]. The Novikov complex is a generalization to the circle - valued case of the Morse complex [5], [7].

The classical Morse theory associates with each Morse function $f : M \to \mathbb{R}$ and a transverse f-gradient the Morse complex. The Novikov theory associates with each circle - valued map $f : M \to S^1$ and a transverse f - gradient the Novikov complex.

In this paper we get new bounds to the number of critical points of circle - valued mappings using Morse - Novikov inequalities for circle - valued functions [6]. We use the $\varphi_{\mathcal{F}}$ - category associated to the family of circle - valued Morse functions defined on a closed manifold M. It is called the Morse-Smale characteristic of manifold M for circle-valued Morse functions and it is denoted by $\gamma_{s^1}(M)$ [1], [2]. The author's results involving Morse -Smale characteristic for circle - valued functions can be found in papers [3] and [4].

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1. Introduction

Recall that in the classical Morse theory, for a compact m-dimensional manifold M and a Morse function $f : M \to \mathbb{R}$ the Morse inequalities are:

$$c_i(f) \geq b_i(M) + q_i(M) + q_{i-1}(M)$$

where $b_i(M)$ are the Betti numbers of M, that is $b_i(M) = dim_{\mathbb{Z}}(H_i(M)/T_i(M))$ and $q_i(M)$ is the minimum number of generators of $T_i(M)$, the torsion part of the homology group $H_i(M)$, $i = 0, 1, \cdots, m$.

Now we will present these relations in the case of circle-valued Morse functions.

2. Morse - Novikov inequalities for circle - valued mappings

The Novikov homology $H_*^{Nov}(M, f, \widehat{\mathbb{Z}[\Pi]})$ is defined for a space M with a map $f : M \to S^1$ and a factorization of $f_* : \pi_1(M) \to \pi_1(S^1)$ through a group Π.

Given a group π and an automorphism $\lambda : \pi \to \pi$, let $\pi \times_\lambda \mathbb{Z}$ be the group with elements gz^i, $g \in \pi$ and $j \in \mathbb{Z}$, and multiplication by $gz = \lambda(g)z$, such that we have the relation:

$$\mathbb{Z}[\pi \times_\lambda \mathbb{Z}] = \mathbb{Z}[\pi]_\lambda[z, z^{-1}].$$

Remark 1. *Consider M to be connected, and let $f : M \to S^1$ be a circle-valued function. The infinite cyclic covering $\bar{M} = f^*(\mathbb{R})$ is connected if and only if $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ is onto, in which case we have:*

$$\pi_1(M) = \pi_1(\bar{M}) \times_{\lambda_M} \mathbb{Z},$$

where $\lambda_M : \pi_1(\bar{M}) \to \pi_1(\bar{M})$ is the automorphism induced by $z : \bar{M} \to \bar{M}$.

If M is connected, with a cohomology class $f \in [M, S^1] = H^1(M)$ such that $\bar{M} = f^*(\mathbb{R})$ is connected, given a factorization of the surjection $f_* : \pi_1(M) \to \pi_1(S^1)$

$$f_* : \pi_1(M) = \pi_1(\bar{M}) \times_{\lambda_M} \mathbb{Z} \to \Pi \to \mathbb{Z}$$

let $\pi = ker(\Pi \to \mathbb{Z})$ and let $z \in \Pi$ be the image of $z = (0, 1) \in \pi_1(M)$, so that $\Pi = \pi \times_\lambda \mathbb{Z}$, with $\lambda : \pi \to \pi$ and $g \to z^{-1}gz$.

The $\widehat{\mathbb{Z}[\Pi]}$ - coefficient Novikov homology of (M, f) is

$$H_*^{Nov}(M, f, \widehat{\mathbb{Z}[\Pi]}) = H_*(M, \widehat{\mathbb{Z}[\Pi]})$$

with $\widehat{\mathbb{Z}[\Pi]} = \mathbb{Z}[\pi_\lambda((z))]$.

In the original case

$$\tilde{M} = \bar{M}\ ,\ \pi = \{1\}\ ,\ \Pi = \mathbb{Z}\ ,\ \widehat{\mathbb{Z}[\Pi]} = \mathbb{Z}((z)),$$

and $H_*^{Nov}(M, f, \widehat{\mathbb{Z}[\Pi]})$ may be written as $H_*^{Nov}(M, f)$ or just $H_*^{Nov}(M)$.

Let $\mathbb{Z}((z))$ be the Novikov ring (a principal domain) and $H_*^{Nov}(M, f)$ the homology of a free $\mathbb{Z}((z))$ - module chain complex.

Definition 1. *The Novikov numbers of any CW-complex M and $f \in H^1(M)$ are $b_i^{Nov}(M,f)$ and $q_i^{Nov}(M,f)$, where*

$$b_i^{Nov}(M,f) = dim_{\mathbb{Z}((z))}(H_i^{Nov}(M,f)/T_i^{Nov}(M,f))$$

are the Betti numbers of Novikov homology and $q_i^{Nov}(M,f)$ is the minimum numbers of generators of $T_i^{Nov}(M,f)$, where

$$T_i^{Nov}(M,f) = \{x \in H_i^{Nov}(M,f) : ax = 0, a \neq 0 \in \mathbb{Z}((z))\}$$

is the torsion $\mathbb{Z}((z))$ - submodule of $H_i^{Nov}(M,f)$.

Proposition 1. *The Novikov complex $C^{Nov}(M,f,v)$ is $\widehat{\mathbb{Z}[\Pi]}$-module equivalent to $C(M;\widehat{\mathbb{Z}[\Pi]})$, with isomorphisms*

$$H_*(C^{Nov}(M,f,v)) \cong H_*^{Nov}(M,f;\widehat{\mathbb{Z}[\Pi]}).$$

The following important relations are called the Morse - Novikov inequalities:

Theorem 1. *For a compact m-dimensional manifold M and a circle-valued Morse function $f : M \to S^1$ the Morse - Novikov inequalities are:*

$$c_i(f) \geq b_i^{Nov}(M,f) + q_i^{Nov}(M,f) + q_{i-1}^{Nov}(M,f),$$

$i = 0, 1, \cdots, m$.

These inequalities are immediate consequence of the isomorphism $H_*(C^{Nov}(M,f,v)) \cong H_*^{Nov}(M,f)$ since for any free chain complex C over a principal domain R we have

$$dim_R(C_i) \geq b_i(C) + q_i(C) + q_{i-1}(C),$$

where $b_i(C) = dim_R(H_i(C)/T_i(C))$, $q_i(C)$ is the minimal number of R-module generators of $T_i(C)$ and

$$T_i(C) = \{x \in H_i(C) : rx = 0, r \neq 0 \in R\}$$

is the R-torsion submodule of $H_i(C)$. The Novikov numbers of M depends only on the cohomology class $\xi = f^*(1) \in H^1(M)$, and so may be denoted by $b_i(M;\mathbb{Z}((z))) = b_i(\xi)$ and $q_i(M;\mathbb{Z}((z))) = q_i(\xi)$.

Theorem 2. *(see [5]) For $\pi_1(M) = \mathbb{Z}$ and $m \geq 6$, let $f : M \to S^1$ be a Morse function, $1 \in [M, S^1] = H^1(M)$ with the minimum numbers of critical points. Then for all $i = 0, 1, \cdots, m$ the following relations hold:*

$$c_i(f) = b_i^{Nov}(M, f) + q_i^{Nov}(M, f) + q_{i-1}^{Nov}(M, f)$$

3.New bounds to the number of critical points of circle - valued mappings

Consider M^m, N^n two smooth manifolds without boundary. For a mapping $f \in C^\infty(M, N)$ denote by $\mu(f) = |C(f)|$, the cardinal number of critical set $C(f)$ of f.

Let $\mathcal{F} \subseteq C^\infty(M, N)$ be a family of smooth mappings $M \to N$.

The $\varphi_{\mathcal{F}}$ - category of the pair (M, N) is defined by

$$\varphi_{\mathcal{F}}(M, N) = \min\{\mu(f) : f \in \mathcal{F}\}.$$

This notion was introduced by D. Andrica in the paper [2]. It is clear that $0 \leq \varphi_{\mathcal{F}}(M, N) \leq +\infty$ and $\varphi_{\mathcal{F}}(M, N) = 0$ if and only if the family $\mathcal{F}$ contains immersions, submersions or local diffeomorphisms according to the cases $m < n, m > n$, or $m = n$, respectively. Under some hypotheses $\varphi_{\mathcal{F}}(M, N)$ is a differential invariant of pair (M, N).(see [1] and [2]).

Using these notations, in the papers [3] and [4] we have considered $N = S^1$ and the family $\mathfrak{F} \subseteq C^\infty(M, S^1)$, given by the set of all circle-valued Morse functions defined on M.

In this case we obtain $\varphi_{\mathcal{F}}(M, S^1) = \gamma_{S^1}(M)$, the Morse-Smale characteristic of manifold M for circle-valued Morse functions $f : M \to S^1$. So, we have

$$\gamma_{S^1}(M) = \min\{\mu(f) : f \in \mathcal{F}(M, S^1)\}.$$

In what follows we will use the Morse-Novikov inequalities to get a lower bound to $\gamma_{S^1}(M)$.

Let $f : M \to S^1$ be a circle-valued Morse function, and let $f^* : H^1(S^1) \to H^1(M)$ be the induced homomorphism in cohomology. Denote

$$F^1(M) = \{f^*(1) : f \in \mathcal{F}(M, S^1)\} \subseteq H^1(M).$$

Theorem 3. *The following inequality holds:*

$$\gamma_{S^1}(M) \geq min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\},$$

where $b^{Nov}(\xi) = \sum_{i=0}^{m} b_i^{Nov}(\xi)$ *is the total Betti number of manifold* M *with respect to the cohomology class* $\xi \in H^1(M)$.

Proof. Let $f : M \to S^1$ be a circle-valued Morse function.

Applying the Morse - Novikov inequalities we get:

$$c_i(f) \geq b_i^{Nov}(\xi) + q_i^{Nov}(\xi) + q_{i-1}^{Nov}(\xi),$$

$i = 0, 1, \cdots, m$, hence

$$\mu(f) = \sum_{i=0}^{m} c_i(f) \geq \sum_{i=0}^{m} (b_i^{Nov}(\xi) + q_i^{Nov}(\xi) + q_{i-1}^{Nov}(\xi))$$

$$= b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi)$$

$$\geq min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\}.$$

Taking into account that $f : M \to S^1$ is an arbitrary circle-valued Morse function, it follows that $\min\{\mu_{S^1}(f) = |C_{S^1}(f)| : f \in \mathcal{F}(M, S^1)\} \geq min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\}$, and we are done.

□

Theorem 4. *For* $\pi_1(M) = \mathbb{Z}$ *and* $m \geq 6$, *the following relation holds:*

$$\gamma_{S^1}(M) = min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\}.$$

Proof. Let $f : M \to S^1$ be a circle-valued Morse function with minimal number of critical points. From Theorem 2 it follows that

$$c_i(f) = b_i^{Nov}(\xi) + q_i^{Nov}(\xi) + q_{i-1}^{Nov}(\xi),$$

$i = 0, 1, \cdots, m$, hence $\gamma_{S^1}(M) \leq \mu(f) = \sum_{i=0}^{m} c_i(f) = \sum_{i=0}^{m} (b_i^{Nov}(\xi) + q_i^{Nov}(\xi) + q_{i-1}^{Nov}(\xi)) = b(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi)$, that is $\gamma_{S^1}(M) \leq min\{b^{Nov}(\xi) + q_m^{Nov}(\xi) + 2\sum_{i=0}^{m-1} q_i^{Nov}(\xi) : \xi \in F^1(M)\}$. Taking into the inequality in Theorem 3 the desired result follows.

□

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