GRAPHS WITH F-SYMMETRIC INDEPENDENCE POLYNOMIALS

Vadim E. Levit and Eugen Mandrescu

Abstract. An *independent* set in a graph is a set of pairwise non-adjacent vertices, and $\alpha(G)$ is the size of a maximum independent set in the graph G.

If s_k is the number of independent sets of cardinality k in G, then

$$I(G;x) = s_0 + s_1 x + s_2 x^2 + ... + s_\alpha x^\alpha, \alpha = \alpha(G),$$

is called the *independence polynomial* of G (I. Gutman and F. Harary, 1983).

If $s_{\alpha-i} = f(i) \cdot s_{\alpha-j}$ holds for every $i \in \{0, 1, ..., \lfloor \alpha/2 \rfloor\}$, then $I(G;x)$ is called *f-symmetric*. The *corona* of the graphs G and H is the graph $G \circ H$ obtained by joining each vertex of G to all the vertices of a copy of H.

In this paper we show that for every graph G, the independence polynomial of $G \circ (K_p \cup K_q)$ is f-symmetric, where

$$f(i) = (pq)^{\frac{\alpha}{2}-i}, 0 \leq i \leq \left\lfloor \frac{\alpha}{2} \right\rfloor, \alpha = \alpha(G \circ (K_p \cup K_q)).$$

In particular, we deduce a result of Stevanović [20], claiming that $I(G \circ 2K_1; x)$ is symmetric, i.e., $s_{\alpha-i} = s_{\alpha-j}$ holds for every $i \in \{0, 1, ..., \lfloor \alpha(G \circ 2K_1)/2 \rfloor\}$.

Key words: independent set, independence polynomial, symmetric polynomial.

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1. Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X.

By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V \ \ \textit{and} \ vw \in E\}$, and $N_G[v] = N_G(v) \cup \{v\}$; if there is no ambiguity on G, we use $N(v)$ and $N[v]$, respectively. K_n, P_n, C_n denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the chordless cycle on $n \geq 3$ vertices.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1)$, $V(G_2)$, and as edge set the disjoint union of $E(G_1)$, $E(G_2)$. In particular, nG denotes the disjoint union of $n > 1$ copies of the graph G.

The *Zykov sum* of the disjoint graphs G_1, G_2 is the graph $G_1 + G_2$ with $V(G_1) \cup V(G_2)$ as a vertex set and

$$E(G_1) \cup E(G_2) \cup \{v_1 v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$$

as an edge set [22].

The *corona* of the graphs G and H is the graph $G \circ H$ obtained from G and $|V(G)|$ copies of H, such that each vertex of G is joined to all vertices of a copy of H [3].

An *independent* (or a *stable*) set in G is a set of pairwise non-adjacent vertices. An independent set of maximum size will be referred to as a *maximum independent set* of G, and the *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set in G, and $\omega(G) = \alpha(\overline{G})$, where $\overline{G}$ is the complement of G.

Let s_k be the number of independent sets of size k in a graph G. The polynomial

$$I(G;x) = s_0 + s_1 x + s_2 x^2 + ... + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of G [4]. For a survey on independence polynomials of graphs, see [12].

Independence polynomial was defined as a generalization of matching polynomial of a graph, because the matching polynomial of a graph G and the independence polynomial of its line graph are identical. Recall that given a graph G, its *line graph* $L(G)$ is the graph whose vertex set is the edge set of G, and two vertices are adjacent if they share an end in G.

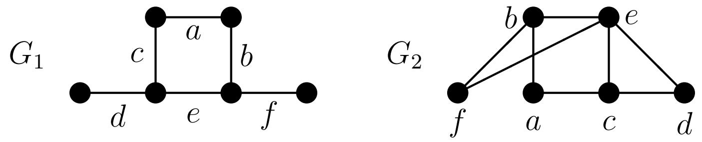

Figure 1: G_2 is the line-graph of and G_1.

For instance, the graphs G_1 and G_2 depicted in Figure 1 satisfy $G_2 = L(G_1)$ and, hence

$$I(G_2; x) = 1 + 6x + 7x^2 + x^3 = M(G_1; x),$$

where $M(G_1; x)$ is the matching polynomial of the graph G_1. Some basic procedures to compute the independence polynomial of a graph are recalled in the following result.

Theorem 1 *[4]* (i) $I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x)$;
(ii) $I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$;
(iii) $I(G; x) = I(G - v; x) + x \cdot I(G - N[v]; x)$ *holds for every* $v \in V(G)$.

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be:

- *unimodal* if there exists an index $k \in \{0, 1, ..., n\}$, called the *mode* of the sequence, such that

$$a_0 \leq ... \leq a_{k-1} \leq a_k \geq a_{k+1} \geq ... \geq a_n;$$

- *log-concave* if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for $i \in \{1, 2, ..., n-1\}$;
- *f-symmetric* if $a_{n-i} = f(i) \cdot a_i$ for all $i \in \{0, ..., \lfloor n/2 \rfloor\}$;
- *symmetric* (or *palindromic*) if $a_i = a_{n-i}, i = 0, 1, ..., \lfloor n/2 \rfloor$, i.e., $f(i) = 1$ for all $i \in \{0, ..., \lfloor n/2 \rfloor\}$.

It is known that every log-concave sequence of positive numbers is also unimodal.

A polynomial is called *unimodal (log-concave, symmetric, f-symmetric)* if the sequence of its coefficients is unimodal (log-concave, symmetric, and f-symmetric, respectively).

Alavi, Malde, Schwenk and Erdös [1] proved that for every permutation π of $\{1, 2, ..., \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that

$$s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)}.$$

For instance, the independence polynomial

- $I(K_{42} + 3K_7; x) = 1 + 63x + 147x^2 + 343x^3$ is log-concave;
- $I(K_{43} + 3K_7; x) = 1 + 64x + 147x^2 + \mathbf{343}x^3$ is unimodal, but non-log-concave, because $147 \cdot 147 - 64 \cdot 343 = -343 < 0$;
- $I(K_{127} + 3K_7; x) = 1 + \mathbf{148}x + 147x^2 + \mathbf{343}x^3$ is non-unimodal;
- $I(K_{18} + 3K_3 + 4K_1; x) = 1 + 31x + \mathbf{33}x^2 + 31x^3 + x^4$ is symmetric and unimodal;
- $I(K_{52} + 3K_4 + 4K_1; x) = 1 + 68x + \mathbf{54}x^2 + 68x^3 + x^4$ is symmetric and non-unimodal;
- $I(K_{1832} + 4K_7 + (K_2 \cup K_{539}) + 5K_1; x) = 1 + 2406x + \mathbf{1382}x^2 + \mathbf{1382}x^3 + 2406x^4 + x^5$ is palindromic and non-unimodal.
- $I\left(P_3 \circ (K_2 \cup K_1); x\right) = 1 + 12x + 52x^2 + 105x^3 + 104x^4 + 48x^5 + 8x^6$ is f-symmetric for $f(i) = 2^{3-i}, 0 \leq i \leq 3$.

It is easy to see that:

- if $\alpha(G) \leq 3$ and $I(G; x)$ is symmetric, then it is also log-concave;
- if $\alpha(G) = 4$ and $I(G; x)$ is symmetric and unimodal, then it is log-concave as well.

For other examples, see [1], [9], [10], [11], [13], [15], [19], and [21].

Theorem 2 *[6]* $I\left(G \circ H; x\right) = \left(I\left(H; x\right)\right)^n \bullet I\left(G; \frac{x}{I(H;x)}\right)$, *where* $n = |V\left(G\right)|$.

The symmetry of matching polynomial and characteristic polynomial of a graph were examined in [8], while for independence polynomial we quote [7], [20], [14], [16], and [18].

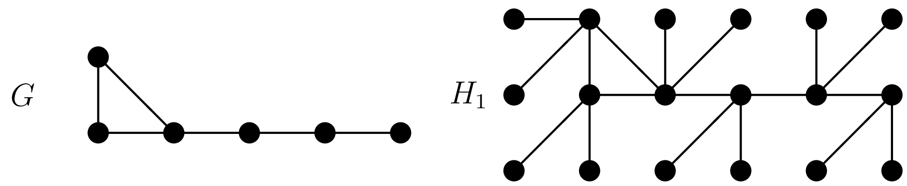

Figure 2: G and $H_1 = G \circ H$, where $H = 2K_1$.

It is worth mentioning that one can produce graphs with symmetric independence polynomials by different ways [2], [5], [20], [18]. For an example, see Figure 2, where $I(G; x) = 1 + 6x + 9x^2 + 2x^3$, while

$$I(H_1; x) = (1 + x)^6 \left(1 + 12x + 48x^2 + 76x^3 + 48x^4 + 12x^5 + x^6\right) =$$

$$= 1 + 18x + 135x^2 + 564x^3 + 1479x^4 + 2586x^5 + 3106x^6 +$$
$$+ 2586x^7 + 1479x^8 + 564x^9 + 135x^{10} + 18x^{11} + x^{12}.$$

In this paper we show that the independence polynomial of the graph $G \circ (K_p \cup K_q)$ is f-symmetric. As a corollary it gives a theorem due to Stevanović claiming that $I(G \circ 2K_1; x)$ is symmetric for every graph G [20].

2. Results

It is well-known that a polynomial $P(x)$ is symmetric if and only if the following equality holds

$$P(x) = x^{\deg(P)} \cdot P\left(\frac{1}{x}\right).$$

Similarly, we have the following.

Lemma 3 *If* $P(x) = \sum_{i=0}^{2n} a_i x^i$ *is a polynomial of degree* $2n$*, then*

$$P(x) = c^n \cdot x^{2n} \cdot P\left(\frac{1}{cx}\right) \text{ if and only if } a_{2n-i} = c^{n-i} \cdot a_i, 0 \leq i \leq n.$$

Proof. Since

$$c^n \cdot x^{2n} \cdot P\left(\frac{1}{cx}\right) = c^n \cdot x^{2n} \cdot \sum_{i=0}^{2n} \frac{a_i}{(cx)^i} = \sum_{i=0}^{2n} c^{n-i} \cdot a_i \cdot x^{2n-i} = \sum_{i=0}^{2n} c^{i-n} \cdot a_{2n-i} \cdot x^i,$$

we infer that

$$P(x) = c^n \cdot x^{2n} \cdot P\left(\frac{1}{cx}\right) \Leftrightarrow a_i = c^{i-n} \cdot a_{2n-i} \Leftrightarrow a_{2n-i} = c^{n-i} \cdot a_i, 0 \le i \le n,$$

and this completes the proof.

Theorem 4 *The polynomial $I(G \circ (K_p \cup K_q); x)$ is f-symmetric, with*

$$f(i) = (pq)^{\frac{\alpha}{2}-i}, \ 0 \le i \le \frac{\alpha}{2}, \ \textit{where } \alpha = \alpha(G \circ (K_p \cup K_q)),$$

i.e., the coefficients (s_i) of $I(G \circ (K_p \cup K_q); x)$ satisfy

$$s_{\alpha-i} = (pq)^{\frac{\alpha}{2}-i} \cdot s_i, \ 0 \le i \le \frac{\alpha}{2}.$$

Proof. Firstly, we have that

$$I(K_p \cup K_q; x) = 1 + ax + bx^2,$$

where $a = p + q$ and $b = pq$.

Secondly, by Theorem 2, we get that

$$I(G \circ (K_p \cup K_q); x) = \left(1 + ax + bx^2\right)^n \cdot I\left(G; \frac{x}{1 + ax + bx^2}\right),$$

where $n = |V(G)|$.

Since each vertex of G is joined, in $G \circ (K_p \cup K_q)$, to all the vertices of a copy of $K_p \cup K_q$, it is clear that

$$\deg I(G \circ (K_p \cup K_q); x) = \alpha(G \circ (K_p \cup K_q)) = 2n.$$

To get the result, we use Lemma 3, i.e., we have to show that

$$\left(1 + ax + bx^2\right)^n \cdot I\left(G; \frac{x}{1 + ax + bx^2}\right) =$$

$$= b^n \cdot x^{2n} \cdot \left(1 + a \cdot \frac{1}{bx} + b \cdot \left(\frac{1}{bx}\right)^2\right)^n \cdot I\left(G; \frac{\frac{1}{bx}}{1 + a \cdot \frac{1}{bx} + b \cdot \left(\frac{1}{bx}\right)^2}\right).$$

Using the fact that

$$\frac{x}{bx^2+ax+1} = \frac{\frac{1}{bx}}{1+a\cdot\frac{1}{bx}+b\cdot\left(\frac{1}{bx}\right)^2}$$

we get that

$$b^n\cdot x^{2n}\cdot\left(1+a\cdot\frac{1}{bx}+b\cdot\left(\frac{1}{bx}\right)^2\right)^n\cdot I\left(G;\frac{\frac{1}{bx}}{1+a\cdot\frac{1}{bx}+b\cdot\left(\frac{1}{bx}\right)^2}\right)=$$

$$=b^n\cdot x^{2n}\cdot\left(\frac{bx^2+ax+1}{bx^2}\right)^n\cdot I\left(G;\frac{x}{bx^2+ax+1}\right)=$$

$$=\left(1+ax+bx^2\right)^n\cdot I\left(G;\frac{x}{1+ax+bx^2}\right),$$

as claimed.

Corollary 5 *[20] $I(G\circ 2K_1;x)$ is symmetric, for every graph G.*

Proof. Taking $p=q=1$ in Theorem 4, we infer that the coefficients (s_i) of $I(G\circ 2K_1;x)$ satisfy

$$s_{\alpha-i}=(pq)^{\frac{\alpha}{2}-i}\cdot s_i=s_i,\ 0\leq i\leq\frac{\alpha}{2},$$

where $\alpha=\alpha(G\circ 2K_1)$. In other words, $I(G\circ 2K_1;x)$ is symmetric.

Corollary 6 *If the coefficients (s_i) of $I(G\circ(K_p\cup K_q);x)$ satisfy*

$$s_i^2\geq s_{i-1}\cdot s_{i+1}, 1\leq i<\alpha(G\circ(K_p\cup K_q))/2,$$

then $I(G\circ(K_p\cup K_q);x)$ is log-concave.

Proof. If n equals the order of G, then $\alpha(G\circ(K_p\cup K_q))=2n$. According to Theorem 4, the coefficients of $I(G\circ(K_p\cup K_q);x)$ satisfy

$$s_{2n-i}=(pq)^{r-i}\cdot s_i, 0\leq i\leq n.$$

Hence we obtain that

$$0\leq s_i^2-s_{i-1}\cdot s_{i+1}=\left((pq)^{i-n}\cdot s_{2n-i}\right)^2-(pq)^{i-1-n}\cdot s_{2n-(i-1)}\cdot(pq)^{i+1-n}\cdot s_{2n-(i+1)}=$$

$$=\left((pq)^{i-n}\right)^2\cdot\left(s_{2n-i}^2-s_{2n-(i-1)}\cdot s_{2n-(i+1)}\right)$$

which implies that $I(G\circ(K_p\cup K_q);x)$ is log-concave.

3. Conclusions

In this paper we have shown that $I(G \circ (K_p \cup K_q); x)$ enjoys some kind of symmetry property, which we called f-symmetry. It seems to be interesting to find other graphs H such that $I(G \circ H; x)$ satisfy similar properties.

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