A GEOMETRICAL STUDY OF THE BEHAVIOR OF SOME DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is to study the stability of some dynamical systems and to make an interpretation from the geometrical point of view. The discrete dynamical systems are represented through equations with differences. These equations are very important in mathematical modeling for recurrence processes with discontinuous evolution - punctual in time or with periodic delay. Next, we make a geometrical study of the trajectories of the solutions which are described by the rhodonea curve.

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1.Introduction

We are presenting solutions for homogeneous equations with differences having constant coefficients, specifying criteria for the stability of the null equation like the condition for the equation's roots to be inside of the unit circle $|z_k| < 1$. By making an analogy with the Routh Hurwitz criterion stating that the roots of the Z_k polynomial have $ReZ_k < 0$ and by using a homographic conformal mapping inside the $|z| < 1$ circle on the $ReZ < 0$ half plane we obtain the asymptotic stability criterion for the difference equations [1,2,3].

2.Stability criteria for dynamic systems represented through equations with differences

The difference equations model mathematically the continuous or discrete systems from automatique, computers, dynamical systems from mechanics, biology, chemistry, physics, economy. The control of stability needs a criterion which is applied on the characteristic polynomial associate - analogy with the Routh Hurwitz criterion: Let be the difference linear equation with k order of multiplicity, $a_j \in R$:

$$a_0x(n+k)+a_1x(n+k+1)+\ldots+a_{k-1}x(n-1)+a_kx(n)=0; E_k(x_n)=0 \quad (1)$$

We control the stability of the null solution $x(n) \equiv 0$.
The characteristic polynomial associate is

$$P(z)=a_0z^k+a_1z^{k-1}+\ldots a_{k-1}z+a_k=0 \quad (2)$$

P having the real roots $z_1, z_2, \ldots, z_p$ (with m order of multiplicity),
$z_1=z_2=\ldots=z_m$ or complex $z_j=\alpha_j+i\beta_j$ with the perturbed solution:

$$\begin{aligned}&x(n)=c_1z_1^n+c_2z_2^n+\ldots+c_pz_p^n+(k+k_1n+\ldots+k_{m-1}n^{m-1})z^m+\\&+\sum_{j=1}^{n}\left(A_j\left|z_j\right|^n\cos(n\arg z_j)+B_j\left|z_j\right|^n\sin(n\arg z_j)\right),\\&p+m+2h=k\end{aligned} \quad (3)$$

The constants from the solution are determined from the initial conditions:

$$x(0)=x_0, x(1)=x_1, \ldots, x(k-1)=x_{k-1} \quad (4)$$

We observe that $\lim\limits_{n\to\infty} x(n)=0 \Leftrightarrow |z_k|<1$.
If the equations is non homogeneous $E_k(x_n)=g_m(n)=ru_m(n)$ where $r \in R$ and $u(n)$ is a polynomial, we are looking for a particular solution $\tilde{u}(n)=r^n\tilde{u}(n)$, where the degree of the polynomial $\tilde{u}$ is n.
If r is a root of the characteristical equation with the multiplicity order j then $\tilde{u}_{m+j}(n)$.
If $g(n)=u(n)\begin{cases}\sin\alpha n\\ \cos\alpha n\end{cases}$ then $\tilde{u}(n)=r^n\left[u_1(n)\cos\alpha n+u_2(n)\sin\alpha n\right]$.
Observation For the dynamical system t with the independent variable we have the equation:

$$a_0x(t+k)+a_1x(t+k-1)+\ldots+a_kx(t)=0 \quad (5)$$

with the solutions $(z_k)^t, t = 1, 2, 3, \ldots, n$.
Example: *The Fibonacci equation:*

$$x(n+2) - x(n+1) - x(n) = 0 \tag{6}$$

$$z^2 - z - 1 = 0; z_{1,2} = \frac{1 \pm \sqrt{5}}{2}; x(0) = 0; x(1) = 1;$$

$$x(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

The number $\Phi = \left(\frac{1+\sqrt{5}}{2}\right)$ is the gold average or the gold section.
The system in the fazes space is $x_{n+1} = y_n, y_{n+1} = x_n + y_n$ with the associate matrix A:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} ax_n + by_n \\ cx_n + dy_n \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the characteristic polynomial

$$\begin{vmatrix} a-z & b \\ c & d-z \end{vmatrix} = \begin{vmatrix} 0-z & 1 \\ 1 & 1-z \end{vmatrix} = 0.$$

$z^2 - z - 1 = 0$ have in the (x, y) plane the trajectory formed with discrete points, which are out from the critical point (of equilibrium O(0,0)) because $|z| = \frac{1+\sqrt{5}}{2} > 1$ and of the origin is an unstable asymptotic knot.

3. The stability criteria

For the stability of the null solution we have the analogous criterions with the continuous systems with constant coefficients. In the phase's plane for the systems with second order we have the trajectories and the critical point classification:

1. If all roots are different and $|z_k| < 1$, then the solution $x(n) \equiv 0$ is stable asymptotic

2. If there is a root with $|z_j| > 1$ and the others with $|z_k| < 1$ then there is instability

3. If the characteristically equation $P(z) = 0$ has simple roots with $|z_k| = 1$ and the others with $|z_k| < 1$ there is simple stability

4. If the characteristical equation $P(z) = 0$ has at least one multiple root with $|z_k| = 1$ there is instability

For the second order differential systems there are the following cases:

1. $z_{1,2} \in R, |z_{1,2}| < 1, O(0,0)$ is an asymptotic stable knot;

2. $|z_{1,2}| = 1, z_1 \neq z_2, O(0,0)$ is a center simple stable ($z_1 = 1, |z_2| < 1, z_2 \in R$);

3. $|z_{1,2}| = 1, z_1 \neq z_2, z_{1,2} = \alpha \pm i\beta$ is a center simple stable, the trajectory describes an ellipse;

4. $z_{1,2} = \alpha \pm i\beta, |z_k| < 1$ is an asymptotic stable focus, the trajectory describes a spiral.

Next we search the conditions to obtain a stability criterion, using the conformal homografically mapping $Z = \frac{z+1}{z-1}$ which transforms punctually $|z| < 1$ in the half plane $ImZ < 0$.
From $z = \frac{Z+1}{1-Z}$, corresponding to the boundary of circle $z = e^{\theta i}$, we obtain the YY' axis (X=0) and, corresponding to $z = 0$, we obtain the $X = -1$ (the inside of the circle is transforming in the left half plane).
Replacing z in characteristical polynomial it is obtained:

$$a_0(Z+1)^k + a_1(Z+1)^{k-1}(1-Z) + \ldots + a_{k-1}(1-Z)^{k-1}(Z+1) + a_k(1-Z)^k = 0, \tag{7}$$

respectively

$$A_0 Z^k + A_1 Z^{k-1} + \ldots + A_{k-1} Z + A_k = 0 \tag{8}$$

for which it is applied the Routh Hurwitz criterion - the condition is to have all the determinants positive [4,5].
Examples:

1. The stability criterion for the first degree equation:
$a_0 x(n+1) + a_1 x(n) = 0, x(n) = 0$ is asymptotic stable if $a_0 > |a_1|$

2. The criterion for the equation of second degree with differences. We are considering the equation:

$$a_0x(n+2)+a_1x(n+1)+a_2x(n)=0$$

$$a_0z^2+a_1z+a_2=0, |z|<1 \Rightarrow z=\frac{Z+1}{1-Z}$$

we have:

$$a_0(Z+1)^2+a_1(Z+1)(1-Z)+a_2(1-Z)^2=0$$

$$(a_0-a_1+a_2)Z^2+(2a_0-2a_2)Z+a_0+a_1+a_2=0$$

$$A_0=a_0-a_1+a_2, A_1=2(a_0-a_2), A_2=a_0+a_1+a_2$$

according to the Routh - Hurwitz criterion results that $A_0, A_1, A_2 > 0$. It results that for the equation with second degree it is sufficient that:
$a_0-a_1+a_2>0, a_0-a_2>0, a_0+a_1+a_2>0$ resulting $|z_{1,2}|<1$.

4. The relative cinematic and the determination of the floral shapes of the rosaceous - topological problems

It is considered the circle of radius R with the center fixed in O, the diameter AOB is rotating with $\theta=\omega t$. Harmonic oscillations $\varphi=\Omega t$ are produced over the point M which is moving on the diameter, under the influence of the elastic force. The circle is fixed on the system X_1OY_1.
The point M is harmonically oscillating on the AB $\overrightarrow{OM}=x\vec{i}=R\cos\Omega t\vec{i}$. The fixed system X_1OY_1 has the axis with the fixed versors $\vec{i}_1, \vec{j}_1$. Versus this system the point M described the trajectories - translations on the $Ox[-R,R]$ axis and rotations with $\theta=\omega t$. We want to find the link between ω and Ω that the trajectories have (3,4,5,6) symmetrical petals (the periodical curves closed).
Composing the movements versus the fixed system we have:

$$\vec{r_1}(M)=\vec{r_0}+\vec{r}(M), \vec{r}_0=0; \vec{r}_1=X_1i_1+Y_1j_1=x\vec{i}$$

$$\vec{i}=\vec{i}_1\cos\theta+\vec{j}_1\sin\theta, z_1=x_1+iy_1=(-1)^kR\cos\Omega te^{i\omega t}$$

For a position k , $\vec{r}_1(k)$ being the peak on the circle R of a regulated polygon with n sides, $k=0,1,\dots,n-1$. The condition for the moment when the point M arrives at the extremity (A,B) to coincide with the fact that A,B become the peaks of a regulated polygon in the k positions: $\theta=\omega t_k=2\pi-k\frac{2\pi}{n}$ and $\varphi=\Omega t_k=k\pi$, n fixed gives us $\frac{\omega}{\Omega}=\frac{2(n-k)}{kn}$ and for the fazes k $Z_1(k)=$

$(-1)^k R \cdot e^{i\omega t_k}$, $k = 0, 1, \ldots, n-1$, is obtained the peaks of the regulated polygons:

$$x_1(k) = (-1)^k R \cos \frac{2\pi k}{n}, y_1(k) = (-1)^k R \sin \frac{2\pi k}{n} \qquad (9)$$

the triangle for n=3, the square for n=4, the hexagon for n=6, the octagon for n=8.

With a continuous line is obtained $x_1 = x_1(t), y_1 = y_1(t)$ for $t \in (0, 2\pi)$, the floral petals.

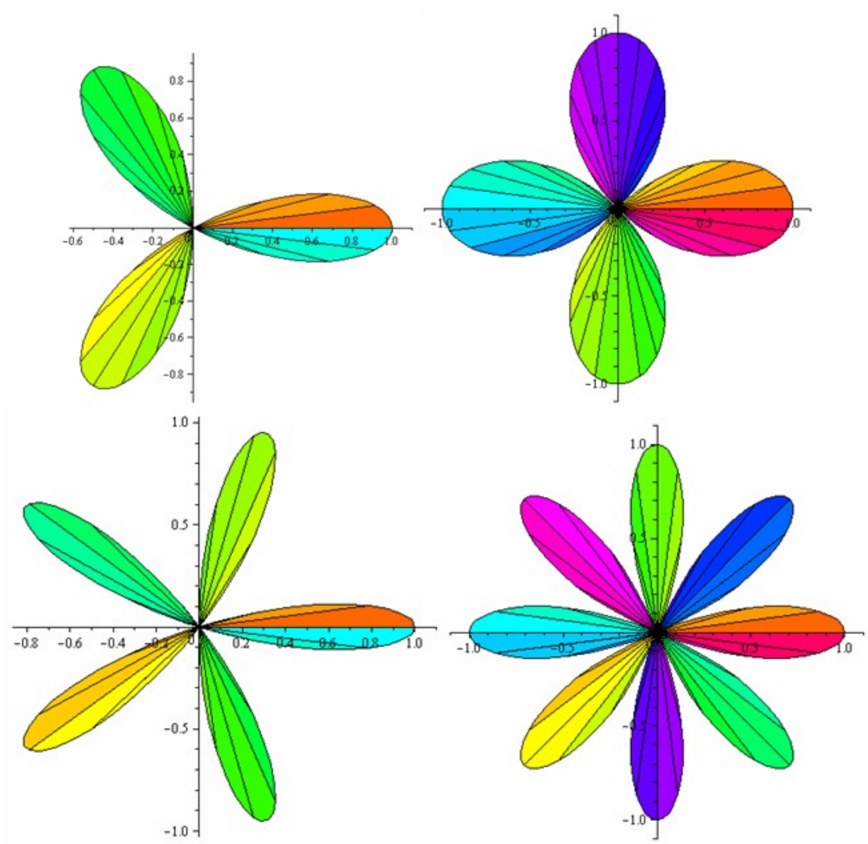

Figure 1: The floral petals for n=3, n=4, n=5; n=8

In this paragraph we have the following problem: if there is a point inside of a circle, it's trajectory goes through every point from inside of circle. If the fraction $\frac{\omega}{\Omega}$ is a rational number the trajectory is periodical and describes a rosaceous with n side; if the fraction is an integer number the trajectory goes through all points of the circle (the set is dense). The study of rosaceous was made using the Maple software.

5. Geometrical properties of a particular rosaceous

The rosaceous (from the Greek: rhodon, rose) is also named Grandi's rose (from Luigi Guido Grandi who studied and named it in 1725). We give some geometrical properties of rosaceous for $n = 3$.

1. The polar representation of the rosaceous, or the equation of the rosaceous in polar coordinates is:

$$(C) \quad \rho = a \sin 3\theta, a \in R_+.$$

The coordinates x, y of a point of rosaceous are:

$$x = \rho \cos \theta, \; y = \rho \sin \theta.$$

Replacing the value of ρ in the expressions for x and y, we obtain the parametric representation of the rosaceous:

$$(C): \; x = a \sin 3\theta \cos \theta, \; y = a \sin 3\theta \sin \theta.$$

Further, eliminating the parameter θ between the two equations, it follows: $\frac{y}{x} = \text{tg}\theta$ and therefore:

$$(C) \; (x^2 + y^2)^2 \pm a(3x^2 y - y^3) = 0,$$

which represents the implicit representation of the rosaceous.

2. The equations of the tangent and the normal line to a rosaceous curve (in polar representation) at an ordinary point are:

$$(T): \; Y - y = \frac{3a \cos 3\theta \text{tg}\theta + a \sin 3\theta}{3a \cos 3\theta - a \sin 3\theta \text{tg}\theta}(X - x),$$

$$(T): \; Y - y = -\frac{3a \cos 3\theta - a \sin 3\theta \text{tg}\theta}{3a \cos 3\theta \text{tg}\theta + a \sin 3\theta}(X - x),$$

where (X, Y) are current coordinates on the tangent/normal lines.

3. The lengths of the tangent segment, subtangent segment, normal segment and subnormal segment of $(C): \; \rho = asin3\theta$ are:

$$s_{\text{tg}} = |\frac{\rho}{\rho'}|\sqrt{\rho^2 + \rho'^2} = \frac{a}{3}|\text{tg}3\theta|\sqrt{9 - 8\sin^2 3\theta},$$

$$s_{stg} = \frac{\rho^2}{|\rho'|} = \frac{a}{3}\frac{sin^2 3\theta}{|cos3\theta|},$$

$$s_n = \sqrt{\rho^2 + \rho'^2} = \sqrt{9 - 8\sin^2 3\theta},$$

$$s_{sn} = |\rho'| = 3a|\cos 3\theta|.$$

4. The rosaceous (C) $(x^2+y^2)^2 \pm a(3x^2y - y^3) = 0$, *is a unicursal curve* (its parametric equations express the coordinates (x, y) of a certain point of the curve, as rational functions of a parameter t):

$$(C)\begin{cases} x = \pm\frac{2at(1-t^2)(t^2-3)(3t^2-1)}{(1+t^2)^4}, \\ y = \pm\frac{4at^2(t^2-3)(3t^2-1)}{(1+t^2)^4}, \end{cases}$$

where $t = \mathrm{tg}\frac{\theta}{2}$, $\mathrm{tg}\theta = \frac{2t}{1-t^2}$, $\frac{y}{x} = \mathrm{tg}\theta$.
6. The multiple points of the rosaceous: (C) $(x^2+y^2)^2 \pm a(3x^2y - y^3) = 0$, are: $O(0,0)$ is a triple point, with the three real and distinct tangent lines having the equations:

$$(T_1):\ y = 0;\ (T_{2,3})\ y = \pm\sqrt{3}x,$$

$A_{1,2}(\pm\frac{3\sqrt{3}a}{8}, \frac{3a}{8})$ and $A_{3,4}(\pm\frac{3\sqrt{3}a}{8}, -\frac{3a}{8})$ are double points. $A_{1,2}$ or $A_{3,4}$ are knots for $a \in (0, \frac{8}{3\sqrt{15}})$, respectively $a \in (0, \frac{8}{\sqrt{7}})$, with the equations of the two real and distinct tangent lines:

$$(T_{1,2}):\ y = \frac{8 \pm \sqrt{64 - 135a^2}}{15a}x \quad (T_{3,4}):\ y = \frac{8 \pm \sqrt{64 - 7a^2}}{a}x.$$

$A_{1,2}$ or $A_{3,4}$ are cusps if $a = \frac{8}{3\sqrt{15}}$, respectively $a = \frac{8}{\sqrt{7}}$, with the equations of the two identical, real tangent lines:

$$(T_{1,2}):\ y = \frac{8}{15a}x \quad (T_{3,4}):\ y = \frac{8}{a}x.$$

$A_{1,2}$ or $A_{3,4}$ are isolated points for $a > \frac{8}{3\sqrt{15}}$, respectively for $a > \frac{8}{\sqrt{7}}$.

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