COMPLETELY REGULAR BICLOSUER SPACES

Halgwrd M. Darwesh and Hardi N. Aziz

Abstract. The purpose of this paper is to introduce the concept of completely regular biclosure spaces and investigate some of their properties.

2000 *Mathematics Subject Classification*: 54A05, 54B05.

1. Introduction

Closure spaces were introduced by Čech [1]. The notion of closure operators are very useful tool in several areas of classical mathematics. They play an important role in topological spaces [1], boolean algebra [2] and digital topology [3]. Kelly [4] introduced the notion of bitopological space. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, so many papers have been written to generalize topological concepts to bitopological setting. Boonpok [5] introduced the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. He extended the concepts of Hausdorffness [6], Regularity [7] and Normality [8] to biclosure setting. In this paper, we introduce and study the concept of completely regular biclosure spaces and some of their properties.

2.Preliminaries

Throughout this paper, I denotes the closed interval $[0,1]$, and $cl_1 = cl_2$ denote the closure operators on I associated to the usual relative topology of I, respectively. So, we recall the following definitions and results from [2,5,6,7,8].

Definition 2.1 *A map $c : P(X) \to P(X)$ defined on a power set $P(X)$ of a set X is called a closure operator on X and the pair (X, c) is called a closure space if the following axioms are satisfied:*
C1) $c\phi = \phi$.
C2) $A \subseteq cA$ for every $A \subseteq X$.
C3) for all A, $B \subseteq X$, if $A \subseteq B$, then $cA \subseteq cB$.
Definition 2.2 *A subset A of a closure space (X, c) is called closed if $cA = A$, and it is open if its complement in X is closed.*
Definition 2.3 *Let (X, c) and (Y, e) be two closure spaces. A function $f : X \to Y$ is said to be continuous if $f(cA) \subseteq ef(A)$ for every subset A of X.*
Definition 2.4 *The product of a family $\{(X_\lambda, c_\lambda) : \lambda \in \Lambda\}$ of closure spaces, denoted by $\prod_{\lambda\in\Lambda}(X_\lambda, c_\lambda)$, is the closure space $(\prod_{\lambda\in\Lambda} X_\lambda, c)$, where $\prod_{\lambda\in\Lambda} X_\lambda$ denotes the cartesian product of sets X_λ, $\lambda \in \Lambda$, and c is the closure operatapor generated by the projections $\pi_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$, $\lambda \in \Lambda$, i.e., is defined by $cA = \prod_{\lambda\in\Lambda} c_\lambda \pi_\lambda(A)$ for each $A \subseteq \prod_{\lambda\in\Lambda} X_\lambda$.*
Definition 2.5 *A biclosure space is a triple (X, c_1, c_2), where X is a set and c_1, c_2 are two closure operators on X.*
Definition 2.6 *A subset A of a biclosure space (X, c_1, c_2) is called closed if $c_1c_2A = A$. The complement of closed set is called open.*
Proposition 2.7 *A subset A of a biclosure space (X, c_1, c_2) is closed if and only if A is closed subset of (X, c_1) and (X, c_2). Furthermore, if A is a closed subset of a biclosure space (X, c_1, c_2), then $c_1c_2A = A$ if and only if $c_1A = A$, $c_2A = A$.*
Definition 2.8 *Let (X, c_1, c_2) be a biclosure space. A biclosure space (Y, e_1, e_2) is called subspace of (X, c_1, c_2) if $Y \subseteq X$ and $e_iA = c_iA \bigcap Y$ for each $i \in \{1, 2\}$ and each $A \subseteq Y$.*
Proposition 2.9 *Let (X, c_1, c_2) be a biclosure space and let (Y, e_1, e_2) be a closed subspace of (X, c_1, c_2). If F is a closed subset of (Y, e_1, e_2), then F is a closed subset of (X, c_1, c_2).*
Proposition 2.10 *Let $\{(X_\lambda, c^1_\lambda, c^2_\lambda) : \lambda \in \Lambda\}$ be a family of biclosure spaces and let $\alpha \in \Lambda$. Then F is a closed subset of $(X_\alpha, c^1_\alpha, c^2_\alpha)$ if and only if $F \times \prod_{\lambda\in\Lambda-\{\alpha\}} X_\lambda$ is a closed subset of $\prod_{\lambda\in\Lambda}(X_\lambda, c^1_\lambda, c^2_\lambda)$.*
Proposition 2.11 *Let $\{(X_\lambda, c^1_\lambda, c^2_\lambda) : \lambda \in \Lambda\}$ be a family of biclosure spaces and let $\alpha \in \Lambda$. Then G is an open subset of $(X_\alpha, c^1_\alpha, c^2_\alpha)$ if and only if $G \times \prod_{\lambda\in\Lambda-\{\alpha\}} X_\lambda$*

is an open subset of $\prod\limits_{\lambda\in\Lambda}(X_\lambda, c^1_\lambda, c^2_\lambda)$.

Definition 2.12 *Let* (X, c_1, c_2) *and* (Y, e_1, e_2) *be two biclosure space and* $i \in \{1, 2\}$. *A function* $f : (X, c_1, c_2) \to (Y, e_1, e_2)$ *is said to be i-continuous if* $f : (X, c_i) \to (Y, e_i)$ *is continuous, and* f *is called continuous if* $f : (X, c_i) \to (Y, e_i)$ *is i-continuous for each* $i \in \{1, 2\}$.

Definition 2.13 *Let* (X, c_1, c_2) *and* (Y, e_1, e_2) *be two biclosure space and* $i \in \{1, 2\}$. *A function* $f : (X, c_1, c_2) \to (Y, e_1, e_2)$ *is said to be i-closed (i-open) if* $f : (X, c_i) \to (Y, e_i)$ *is closed (open), and* f *is called closed (open) if* $f : (X, c_i) \to (Y, e_i)$ *is i-closed (i-open) for each* $i \in \{1, 2\}$.

Definition 2.14 *A biclosure space* (X, c_1, c_2) *is said to be a Hausdorff biclosure space if, whenever* x *and* y *are distinct points of* X *there exists an open subset* U *of* (X, c_1) *and an open subset* V *of* (X, c_2) *such that* $x \in U$, $y \in V$ *and* $U \bigcap V = \phi$.

Definition 2.15 *A biclosure space* (X, c_1, c_2) *is said to be a regualar biclosure space if, for any closed subset* F *of* (X, c_1) *and any point* $x \in X - F$, *there exist disjoint open subsets* U *and* V *of* (X, c_2) *such that* $x \in U$ *and* $F \subseteq V$.

Definition 2.16 *A biclosure space* (X, c_1, c_2) *is said to be a Normal biclosure space if, for every disjoint closed subset* H *of* (X, c_1) *and clsoed subset* K *of* (X, c_2), *there exists a disjoint open subset* U *of* (X, c_1) *and an open subset* V *of* (X, c_2) *such that* $H \subseteq U$ *and* $K \subseteq V$.

3. Completely Regular Biclosure Space

In this section, we introduce the concept of completely regular biclosure spaces and study some of their properties.

Definition 3.1 *A biclosure space* (X, c_1, c_2) *is called a completely regular (brifly, C.R.) biclosure space if, for each point* $x \in X$ *and each closed subset* F *of* (X, c_1) *such that* $x \notin F$, *there exists a continuous function* $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ *such that* $f(x) = 0$ *and* $f(F) = \{1\}$.

Example 3.2 *Let* $X = \{a, b, c\}$ *and define closure operators* c_1 *and* c_2 *on* X *by* $c_iA = A$ *for each* $A \subseteq X$. *Then* (X, c_1, c_2) *is a C.R. biclosure space.*

Proposition 3.3 *Let* (X, c_1, c_2) *be a biclosure space. Then* (X, c_1, c_2) *is a C.R. biclosure space if and only if for each* $x \in X$ *and each open subset* G *of* (X, c_1) *containing* x, *there exists a continuous function* $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ *such that* $f(x) = 0$ *and* $f(y) = 1$, *for each* $y \notin G$.

Proof. Obvious.

Lemma 3.4 *Let f, $g : (X, c) \to (I, cl_1)$ be continuous function. Then $f \mp g : (X, c) \to (I, cl_1)$ is continuos.*
Proof: Let A be any subset of X. Then $(f \mp g)(cA) = f(cA) \mp g(cA) \subseteq cl_1 f(A) \mp cl_1 g(A) = cl_1(f(A) \mp g(A)) = cl_1((f \mp g)(A))$. Hence $f \mp g$ is continuous.
Proposition 3.5 *A biclosure space (X, c_1, c_2) is C.R. if and only if for each $x \in X$ and each closed subset F of (X, c_1) such that $x \notin F$, there exists a continuous function $g : (X, c_1, c_2) \to (I, cl_1, cl_2)$ such that $g(x) = 1$ and $g(F) = \{0\}$.*
Proof: Let (X, c_1, c_2) be a C.R. biclosure space. Let x be any point of X and F be any closed subset of (X, c_1) such that $x \notin F$. Then, there exists a continuous function $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ such that $f(x) = 0$ and $f(F) = \{1\}$. Since the function $h : (X, c_1, c_2) \to (I, cl_1, cl_2)$ given by $h(a) = 1$ for each $a \in X$ is continuous, so by Lemma 3.4, the function $g : (X, c_1, c_2) \to (I, cl_1, cl_2)$ given by $g(x) = h(x) - f(x)$ is continuous, and further, $g(x) = 1$ and $g(F) = \{0\}$.

Conversely, let x be any point of X and let F be any closed subset of (X, c_1) such that $x \notin F$. Then, by hyposthesis, there exists a continuous function $g : (X, c_1, c_2)) \to (I, cl_1, cl_2)$ such that $g(x) = 1$ and $g(F) = \{0\}$. Then, by Lemma 3.4, the function $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ given by $f(x) = h(x) - g(x)$ is continuous and its clear that $f(x) = 0$ and $f(F) = \{1\}$. Hence (X, c_1, c_2) is a C.R. biclosure space.
Theorem 3.6 *Every C.R. biclosuer space is a regular biclosure space.*
Proof: Let (X, c_1, c_2) be a C.R. biclosure space. Let F be any closed subset of (X, c_1) and $x \in X$ such that $x \notin F$. Then, there exists a continuous function $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ such that $f(x) = 0$ and $f(F) = \{1\}$. Since 0 and 1 are distinct points of a Hausdorff biclosure space (I, cl_1, cl_2), then there exists disjoint open subset G of (I, cl_1) (hence of (I, cl_2) {since $cl_1 = cl_2$}) and an open subset U of (X, cl_2) containing 0 and 1, respectively. Then by Defintion 2.12, $f^{-1}(G)$ and $f^{-1}(U)$ are disjoint open subsets of (X, c_2) containing x and F, respectively. Hence (X, c_1, c_2) is a regular biclosure space.
The following example shows that the converse of the above theorem is not true in general.
Example 3.7 [9] *Let $c_1 = c_2$ be the closure operator associated with the topology of the topological space (X, τ), where $X = \{(x, y) \in R^2 : y \geq 0\} \bigcup \{a\}$, and a is a point not in $\{(x, y) \in R^2 : y \geq 0\}$, and τ is defined by; all points (x, y) with $y \geq 0$ are assumed to be isolated. The basic neighborhoods of $(x, 0)$ are conatins $(x, 0)$ and all but finitely many points of the union of two segments*

$I_x = \{(x, y) : 0 \leq y \leq 2\}$ and $I'_x = \{(x + y, y) : 0 \leq y \leq 2\}$. And the basic neighborhoods of the point a have the form $U_n(a) = \{a\} \bigcup \{(x, y) : x \geq n\}$, where $n = 1, 2, ...$ Then in the same line proof of [9], it is easy to prove that the biclosure space (X, c_1, c_2) is regular but not C.R.

The following examples show that the concepts of Hausdorffness and completely regularity of biclosure spaces are independent concepts.

Example 3.8 *Let $X = \{a, b, c\}$ and let c_1 and c_2 be closure operators on X given by $c_1A = X = c_2A$ for all non-empty subset A of X and $c_1\phi = \phi = c_2\phi$. Then it is easy to see that (X, c_1, c_2) is a C.R. non-Hausdorff biclosuer space.*

Example 3.9 *Consider the Smirnov's deleted sequence topology τ on R [10, Example 2.5.5, p. 46], by letting $G \in \tau$ if and only if $G = U - B$ where $B \subseteq A = \{\frac{1}{n} : n = 1, 2, 3, ...\}$ and U is open in the topology on R. Let $C_1 = C_2$ be the closure operetor of this topology. Then (R, C_1, C_2) is a Hausdorff biclosure space but it is not Regular. Hence, by Theorem 3.6, (R, C_1, C_2) is not C.R.*

The following examples show that the concepts of normality and completely regularity of biclosure spaces are independent concepts.

Example 3.10 *Let $c_1 = c_2$ be the closure operator associated to the topology of Niemytzki's Tangent Disc Topology (X, τ^*) [11, Example 82, p. 101]. Then, the biclosure sapce (X, c_1, c_2) is C.R. but it is not Normal.*

Example 3.11 *Let $X = \{a, b\}$ and let c_1 and c_2 be closuer operators on X given by $c_1\phi = \phi = c_2\phi$, $c_1X = c_1\{a\} = X = c_2\{a\} = c_2X$ and $c_1\{b\} = \{b\} = c_2\{b\}$. Then (X, c_1, c_2) is a Normal biclosure space but it is not a regular biclosure space, hence, in view of Theorem 3.6, it is not a C.R. biclosure space.*

Lemma 3.12 *Let (Y, e_1, e_2) be a subspace of a biclosure space (X, c_1, c_2) and let (Z, k_1, k_2) be any biclosure space. If a function $f : (X, c_1, c_2) \to (Z, k_1, k_2)$ is continuous, then the restriction function $f_{/Y} : (Y, e_1, e_2) \to (Z, k_1, k_2)$ of f on Y is continuous.*

Proof: Let A be any subset of Y. Then $A \subseteq X$. Since $f : (X, c_1, c_2) \to (Z, k_1, k_2)$ is continuous, then $f(c_iA) \subseteq k_if(A)$ for each $i \in \{1, 2\}$. Since $e_iA = c_iA \bigcap Y$ for each $i \in \{1, 2\}$, then $f_{/Y}(e_iA) = f(e_iA) = f(c_iA \bigcap Y) \subseteq f(c_iA) \bigcap f(Y) \subseteq f(c_iA) \subseteq k_if(A) = k_if_{/Y}(A)$. Thus $f_{/Y} : (Y, e_1, e_2) \to (Z, k_1, k_2)$ is continuous.

Theorem 3.13 *Every subspace of a C.R. biclosuer space (X, c_1, c_2) is a C.R. biclosuer space.*

Proof: Let (Y, e_1, e_2) be a subspace of a C.R. biclosure space (X, c_1, c_2). Let F be any closed subset of (Y, e_1) and let y be any point of Y such that

$y \in Y - F$, then c_1F is a closed subset of (X, c_1) in which $y \notin c_1F$. Since (X, c_1, c_2) is a C.R. biclosure space, then there exists a continuous function $f : (X, c_1, c_2) \to (I, cl_1, cl_2)$ such that $f(y) = 0$ and $f(c_1F) = \{1\}$. Now, by Lemma 2.11, the function $f_{/Y} : (Y, e_1, e_2) \to (I, cl_1, cl_2)$ is conntinuous. Further, $f_{/Y}(y) = f(y) = 0$ and for each $f_{/Y}(F) \subseteq f(c_1F) = \{1\}$. This proved that (Y, e_1, e_2) is a C.R. biclosure space.

Lemma 3.14 *Let $f : (X, c_1, c_2) \to (Y, e_1, e_2)$ and $g : (Y, e_1, e_2) \to (Z, k_1, k_2)$ be two continuous functions. Then $g \circ f : (X, c_1, c_2) \to (Z, k_1, k_2)$ is continuous.*

Proof: Obvious.

Theorem 3.15 *Let (X, c_1, c_2) and (Y, e_1, e_2) be two biclosure spaces, and let $g : (X, c_1, c_2) \to (Y, e_1, e_2)$ be an injective, closed and continuous function. If (Y, e_1, e_2) is C.R., then (X, c_1, c_2) is C.R. also.*

Proof: Let F be any closed subset of (X, c_1) and let x be any point of X such that $x \notin F$. Since g is closed, then $g(F)$ is a closed subset of (Y, e_1), and since g is injective, then $g(x) \notin g(F)$. Since (Y, e_1, e_2) is a C.R. biclosure space, then there exists a continuous function $f : (Y, e_1, e_2) \to (I, cl_1, cl_2)$ such that $f(g(x)) = 0$ and $f(g(F)) = \{1\}$. So, by Lemma 3.14, the function $f \circ g : (X, c_1, c_2) \to (I, cl_1, cl_2)$ is continuous, $f \circ g(x)) = 0$ and $f \circ g(F)) = \{1\}$. Hence (X, c_1, c_2) is a C.R. biclosure space.

Theorem 3.16 *Let $\{(X_\lambda, c^1_\lambda, c^2_\lambda) : \lambda \in \Lambda\}$ be a family of biclosure spaces. Then $\prod_{\lambda\in\Lambda} (X_\lambda, c^1_\lambda, c^2_\lambda)$ is a C.R. biclosure space if and only if $(X_\lambda, c^1_\lambda, c^2_\lambda)$ is a C.R. biclosure space for each $\lambda \in \Lambda$.*

Proof: Suppose that $\prod_{\lambda\in\Lambda} (X_\lambda, c^1_\lambda, c^2_\lambda)$ is a C.R. biclosure space, and we assume $\alpha \in \Lambda$. To show $(X_\alpha, c^1_\alpha, c^2_\alpha)$ is C.R.. We choose $x^*_\lambda \in X_\lambda$ for each $\lambda \in \Lambda - \{\alpha\}$. Then, by Theorem 3.13, the subspace $X = (X_\alpha, c^1_\alpha, c^1_\alpha) \times \prod_{\lambda\in\Lambda-\{\alpha\}} (\{x^*_\lambda\}, c^{*1}_\lambda, c^{*2}_\lambda)$ of $\prod_{\lambda\in\Lambda} (X_\lambda, c^1_\lambda, c^2_\lambda)$ is a C.R. biclosure space. Also, it is easy to see that the function $g : (X_\alpha, c^1_\alpha, c^2_\alpha) \to X$ given by $g(x_\alpha) = (x_\lambda)_{\lambda\in\Lambda}$, for each $x_\alpha \in X_\alpha$ and $x_\lambda = x^*_\lambda$, for each $\lambda \in \Lambda - \{\alpha\}$, is a bijective closed and continuous function. Therefore, by Theorem 3.15, $(X_\alpha, c^1_\alpha, c^2_\alpha)$ is C.R.

Conversely, let $x = (x_\lambda)_{\lambda\in\Lambda}$ be any point of $\prod_{\lambda\in\Lambda} X_\lambda$ and let F be any closed subset of $\prod_{\lambda\in\Lambda} (X_\lambda, c^1_\lambda)$ such that $x \notin F$. Then $\pi_\lambda(F)$ is a closed subset of (X_λ, c^1_λ) for each $\lambda \in \Lambda$ and there exists $\alpha \in \Lambda$ such that $x_\alpha \notin \pi_\alpha(F)$. Since $(X_\alpha, c^1_\alpha, c^2_\alpha)$ is C.R., then there exists a continuous function $f : (X_\alpha, c^1_\alpha, c^2_\alpha) \to (I, cl_1, cl_2)$ such that $f(x_\alpha) = 0$ and $f(\pi_\alpha(F)) = \{1\}$. Thus, the function

$f \circ \pi_\alpha : \prod_{\lambda \in \Lambda} (X_\lambda, c^1_\lambda, c^2_\lambda) \to (I, cl_1, cl_2)$ is continuous, $f \circ \pi_\alpha((x_\lambda)_{\lambda \in \Lambda}) = f(x_\alpha) = 0$ and $f \circ \pi_\alpha((F)) = \{1\}$. Hence, $\prod_{\lambda \in \Lambda} (X_\lambda, c^1_\lambda, c^2_\lambda)$ is a C.R. biclosure space.

References

[1] E. Čech, Topological Spaces, Topological Papers of Eduard Čech, Academia, Paraguc, (1968): 436 - 472.
[2] A. S. Mashhour, On Closure Spaces, Indian J. Pure Appl. Math. 14, 6, (1983): 680 - 691.
[3] J. Šlapal, Closure Operations for Digital Topology, Theoret. Comput. Sci., 305, (2003): 457 - 471.
[4] J. C. Kelly, Bitopological Spaces, Pro. London Math. Soc., 3, 13, (1969): 71 - 79.
[5] C. Boonpok, On Pairwise Bicontinuous Map., Int. J. Math. Analy. 4, 12 (2010): 555 - 564.
[6] C. Boonpok, Hausdorff Biclosure Spaces, Int. J. Contemp. Math. Sci., 5, 8, (2010): 359 - 362.
[7] C. Boonpok, Regular Biclosure Spaces, Int. J. Contemp. Math. Sci., 5, 8, (2010): 365 - 371.
[8] C. Boonpok, Normal Biclosure Spaces, Int. Math Forun., 5, 8, (2010): 401 - 407.
[9] A. Mysior, A Regualar Space Which is Not Completely Regular, Amer. Math. soci., 81, 4, (1981): 652 - 653.
[10] T. A. Sunitha. A Study of Črch Closure Spaces, Ph.D Thesis, Co Chin University, College of Science and Technology, 1994.
[11] I. L. Steen, and J. A. Jr Seebanch, Counterexamples in topology, Springer-Verlag, New York. 1978.

Halgwrd M. Darwesh
Department of Mathematics
College of Science
University of Sulaimani
Iraq- Kurdistan Region
email:*darweshymath@yahoo.com*

Hardi N. Aziz
Department of Mathematics
College of Science Education
University of Sulaimani
Iraq- Kurdistan Region
email:*harde1984@yahoo.com*