ON THE MOMENTS OF NORMAL PROBABILITY DISTRIBUTION

by
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Abstract: In this paper the author present a method for finding an explicit expression of the $(r_1, r_2)^{th}$ moments of the two-dimensional random vector having the normal distribution. Also is establish the recurrence relation (12) for the central moments of the order (r_1, r_2) of the random vector (Y_1, Y_2).

1.Consider a two-dimensional random vector $Y = (Y_1, Y_2)$, and let be $F(y_1, y_2; x_1, x_2)$, the probability distribution of (Y_1, Y_2), where (y_1, y_2) is any point of the Euclidian space R^2 and (x_1, x_2) is a real two-dimensional parameter, varying in a parameter space Ω_2, which is a subset of R^2.

Let $f = f(y_1, y_2)$ be a real-valued function defined and bounded on R^2 such that the mean value of the random variable $f(Y_1, Y_2)$ exists.

As is well known, this mean value can be expressed by the improper Stieltjes integral of (y_1, y_2) with respect to $F(y_1, y_2; x_1, x_2)$.

$$(1)\ E[f(Y_1, Y_2)] = \int_{R^2} f(y_1, y_2)\,dF(y_1, y_2; x_1, x_2)$$

Assuming that the random vector (Y_1, Y_2) is of continuous type, having probability densities $\rho(y_1, y_2; x_1, x_2)$, then we have

$$F(y_1, y_2; x_1, x_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} \rho(u_1, u_2; x_1, x_2)\,du_1 du_2$$

and the mean value by (1) is

$$(2)\ E[f(Y_1, Y_2)] = \int_{R^2} f(y_1, y_2)\rho(y_1, y_2; x_1, x_2)\,dy_1 dy_2$$

Considering that

$$f(y_1, y_2) = (y_1 - a_1)^{r_1}(y_2 - a_2)^{r_2}$$

where $r_1, r_2 \in N$ and $(y_1, y_2) \in R^2$, we have the moment of order (r_1, r_2), of the point (a_1, a_2) define if it exists- such the mean value of vectors

$$f(Y_1, Y_2) = (Y_1 - a_1)^{r_1} (Y_2 - a_2)^{r_2}$$

$$(3)\ \nu_{r_1, r_2}(a_1, a_2) = E\left[(Y_1 - a_1)^{r_1} (Y_2 - a_2)^{r_2}\right].$$

For $(a_1, a_2) = (0,0)$, we have the ordinary moment of order (r_1, r_2) of the random vector (Y_1, Y_2).

$$(4)\ \nu_{r_1, r_2}(0,0) = \nu_{r_1, r_2} = E\left[Y_1^{r_1} Y_2^{r_2}\right]$$

If $\nu_{1,0} = E(Y_1)$ and $\nu_{0,1} = E(Y_2)$ we can write expression for the central moment of the order (r_1, r_2) of the random vector (Y_1, Y_2).

$$(5)\ \mu_{r_1, r_2} = E\left[(Y_1 - \nu_{1,0})^{r_1} (Y_2 - \nu_{0,1})^{r_2}\right].$$

2. Let us assume that random variables Y_1, Y_2 are not independent and the two-dimensional random vector Y having the normal distribution with the probability density

$$(6)\ \begin{aligned} \rho(y_1, y_2; x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \\ &\cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1 - x_1}{\sigma_1}\right)^2 - 2\rho\frac{(y_1 - x_1)(y_2 - x_2)}{\sigma_1\sigma_2} + \left(\frac{y_2 - x_2}{\sigma_2}\right)^2\right]\right\} \end{aligned}$$

where $x_1, x_2 \in R$, $\sigma_1 > 0, \sigma_2 > 0, \rho \in [-1,1]$, and $(y_1, y_2) \in R^2$.

It is easy to see that $\rho(y_1, y_2; x_1, x_2) > 0$, $(y_1, y_2) \in R^2$ and

$\int_{R^2} \rho(y_1, y_2; x_1, x_2) dy_1 dy_2 = 1$, if we change the variables

$$(7)\ \begin{cases} y_1 = u\sigma_1 + x_1 \\ y_2 = u\sigma_2 + x_2 \end{cases}$$

with the Jacobian

$$\frac{D(y_1, y_2)}{D(u, v)} = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \sigma_1\sigma_2 > 0$$

For $r_1 = r_2 = 1$, the central moment of the random vector (Y_1, Y_2) become

$$\mu_{1,1} = E[(Y_1 - \nu_{1,0})(Y_2 - \nu_{0,1})] = \int_{R^2} (y_1 - x_1)(y_2 - x_2)\rho(y_1, y_2; x_1, x_2)dy_1dy_2$$

By using the formula (6) we get

$$\mu_{1,1} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \int_{R^2} (y_1 - x_1)(y_2 - x_2)\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1 - x_1}{\sigma_1}\right)^2 - \right.\right.$$

$$\left.\left. - 2\rho\frac{(y_1 - x_1)(y_2 - x_2)}{\sigma_1\sigma_2} + \left(\frac{y_2 - x_2}{\sigma_2}\right)^2\right]\right\}dy_1dy_2$$

According to the relation (7) we have

$$\mu_{1,1} = \frac{\sigma_1\sigma_2}{2\pi\sqrt{1-\rho^2}} \int_R ve^{\frac{-v^2}{2}}\left(\int_R ue^{\frac{-(u-\rho v)^2}{2(1-\rho^2)}}du\right)dv$$

For $u - \rho v = t\sqrt{1-\rho^2}$ we get

(8) $\mu_{1,1} = \rho\sigma_1\sigma_2$

We observe that the parameter ρ is the corelation coefficient of random variables Y_1 and Y_2.

For the ordinary moment of order (1,1) we have

$$\nu_{1,1} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot$$

$$\cdot \int_{R^2} y_1y_2 \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1 - x_1}{\sigma_1}\right)^2 - 2\rho\frac{(y_1 - x_1)(y_2 - x_2)}{\sigma_1\sigma_2} + \left(\frac{y_2 - x_2}{\sigma_2}\right)^2\right]\right\}dy_1dy_2$$

and using relation (7) results

$$\nu_{1,1} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{R^2} (u\sigma_1 + x_1)(v\sigma_2 + x_2)\exp\left[-\frac{v^2}{2} - \frac{(u-\rho v)^2}{2(1-\rho^2)}\right]dudv \cdot$$

If $u-\rho v = t\sqrt{1-\rho^2}$ we get

(9) $\nu_{1,1} = \rho\sigma_1\sigma_2 + x_1x_2$.

3. We should remark that in the special case $\rho = 0$ results $\mu_{1,1} = 0$ and the random variables Y_1 and Y_2 are independent.

In this case ($\rho = 0$) for $r_1 = r_2 = 1$, the ordinary moment of the random vector (Y_1, Y_2) become

$$\nu_{1,1} = E[Y_1Y_2] = \frac{1}{2\pi\sigma_1\sigma_2}\int_{R^2} y_1y_2 \exp\left[-\frac{(y_1-x_1)^2}{2\sigma_1^{\ 2}} - \frac{(y_2-x_2)^2}{2\sigma_2^{\ 2}}\right]dy_1dy_2$$

and using relation (7) we have

$$(10)\ \nu_{1,1} = \frac{1}{2\pi}\int_{R^2}(u\sigma_1 + x_1)(v\sigma_2 + x_2)e^{-\frac{u^2}{2}-\frac{v^2}{2}}dudv = x_1x_2 .$$

Analogous we have

$$\mu_{2,2} = \frac{1}{2\pi\sigma_1\sigma_2}\int_{R^2}(y_1-x_1)^2(y_2-x_2)^2 \exp\left[-\frac{(y_1-x_1)^2}{2\sigma_1^{\ 2}} - \frac{(y_2-x_2)^2}{2\sigma_2^{\ 2}}\right]dy_1dy_2$$

and using the relation (7) we get

$$(11)\ \mu_{2,2} = \sigma_1^2\sigma_2^2 = (\sigma_1\sigma_2)^2$$

It should be noticed that for r_1 or r_2 being a non-negative impare integer we have $\mu_{2k+1,2k+1} = 0$ where $k_1 \in N^*, k_2 \in N^*$ we get recurrence formula for the central moments

$$(12)\ \mu_{r_1,r_2} = (r_1-1)(r_2-1)\sigma_1^2\sigma_2^2\mu_{r_1-2,r_2-2}$$

For the special case $r_1 = r_2 = 2k, k \in N^*$ we have

(13) $\mu_{2k,2k} = (2k-1)^2(2k-3)^2 \cdot \ldots \cdot 3^2 \cdot 1^2 \cdot (\sigma_1\sigma_2)^{2k}, k \in N^*$

4. If , we know the probability density of random variables Y_1, Y_2 and (Y_1, Y_2) we can write the conditional random variables $Y_1|y_2$ and $Y_2|y_1$ with the probability density

$$\rho(y_1|y_2)=\frac{\rho(y_1,y_2;x_1,x_2)}{\rho(y_2;x_2)}=$$

(14)

$$=\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_1-x_1}{\sigma_1}-\rho\frac{y_2-x_2}{\sigma_2}\right)^2\right]$$

and

$$\rho(y_2|y_1)=\frac{\rho(y_1,y_2;x_1,x_2)}{\rho(y_1;x_1)}=$$

(15)

$$=\frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{y_2-x_2}{\sigma_2}-\rho\frac{y_1-x_1}{\sigma_1}\right)^2\right]$$

where $\rho(y_1;x_1)$ is the probability density of random variable Y_1

$$(16)\quad \rho(y_1;x_1)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left[-\frac{(y_1-x_1)^2}{2\sigma_1^2}\right]$$

and $\rho(y_2;x_2)$ is the probability density of random variable Y_2

$$(17)\quad \rho(y_2;x_2)=\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left[-\frac{(y_2-x_2)^2}{2\sigma_2^2}\right]$$

We find that

$$(18)\quad E(Y_1|y_2)=\int_R y_1\rho(y_1|y_2)dy_1 = x_1+\rho\frac{\sigma_1}{\sigma_2}(y_2-x_2)$$

and

$$(19)\quad E(Y_2|y_1)=\int_R y_2\rho(y_2|y_1)dy_2 = x_2+\rho\frac{\sigma_2}{\sigma_1}(y_1-x_1)$$

It should be noticed that the regresion between the random variables Y_1 and Y_2 for the two-dimensional random vector $Y=(Y_1,Y_2)$ it is a liniary function.

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