TANGENT VECTORS TO A DIFFERENTIABLE MANIFOLD IN ONE OF ITS POINTS

by
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Abstract. In this paper the set of tangent vectors to a differentiable manifold in one of its points is defined, some propreties are pointed out and some examples are given in the second part.

1. PRELIMINARY RESULTS

Let (M, $\mathbf{R}^n$, $A_M = \{ (U_\alpha, \varphi_\alpha) \mid \alpha \in I \}$) a n-dimensional differentiable manifold, and let the terns (x, h_a, (X^i)) where $h_a = (U_a, \varphi_a) \in A_M$, $x \in U_a$, $X^i \in \mathbf{R}^n$.

Definition 1.1 On the set of those terns we will define the next relation:

$$(x, h_a, (X^i)) \sim (y, h_b, (Y^i))$$

if and only if:

i) $x = y$

ii) $Y^i = \left(\frac{\partial y^i}{\partial x^j} \right)_x X^j$

where $y^i = y^i (x^1, ..., x^n)$, i = 1, …, n is the coordinates transformation from the map h_a to the map h_b.

It can easily prove that the relation wrote above is an equivalence relation and we will note the equivalence class corresponding the tern (x, h_a, (X^i)) by [(x, h_a, (X^i))], or by X_x and we will call it tangent vector in x to the differentiable manifold M. The numbers (X^i) are called the components of the vector X_x in the map h_a. In the map h_a the vectors X_x can be also write like that:

$$X_x = X^i \left(\frac{\partial}{\partial x^i} \right)_x .$$

We will note by T_xM the set of tangent vectors in x to the differentiable manifold M.

Definition 1.2 A tangent vector in x to M is an equivalence class formed by the curves:

$$c : I \subset \mathbf{R} \to M ;$$
$$0 \in I ;$$

$$c(0) = x$$

in rapport with the equivalence relation:

$$c \sim \gamma$$

if and only if in a map $h_a = (U_a, \varphi_a) \in A_M$, $x \in U_a$ we have:

$$(\varphi_a \circ \gamma)'(0) = (\varphi_a \circ c)'(0).$$

Remark. The definitions 1.1 and 1.2 are equivalent.

Proof. In the definition 1.1 a vector $X_x \in T_xM$ is represented in a local map $h_a = (U_a, \varphi_a)$ by a vector $v \in \mathbf{R}^n$, and in the definition 1.2 by a curve c. we will obtain the equivalence of those two definitions taking:

$$v = (\varphi_a \circ c)'(0)$$

Proposition 1.1 The set of tangent vectors in x to the differentiable manifold M, note by T_xM, is a **R**-vector space, and hence:

$$\dim T_xM = n.$$

Proof. It can easily prove that T_xM is a vector space reported to the following operations:

$$[(x, h_a, (X^i))] + [(x, h_a, (Y^i))] \overset{\text{def}}{=} [(x, h_a, (X^i + Y^i))]$$

$$\alpha [(x, h_a, (X^i))] \overset{\text{def}}{=} [(x, h_a, (\alpha X^i))].$$

Proposition 1.2 If M is a n-dimensional differentiable manifold with the atlas A_M, then $TM = \cup T_xM$ is a 2n-dimensional differentiable manifold.

Proof. If $A_M = \{ h = (U, \varphi) \}$ then $A_{TM} = \{ h_t = (TU, \varphi_{TU}) \}$where $TU = \{ x_n \mid n \in U \}$ and $\varphi_{TU}(x_n) = (\varphi, (x^i))$.

The change of the map is given by:

$$y^i = y^i(x^1, ..., x^n)$$

$$Y^i = \frac{\partial y^i}{\partial x^j} X^j.$$

Definition 1.3 Let M, N being n-dimensional differentiable manifolds and $f : M \to N$ a smooth application. We will define:

$$T_xf : T_xM \to T_{f(x)}N$$

$$T_xf([c]_x) \overset{\text{def}}{=} [c \circ f]_x$$

or, equivalent:

$$T_xf : v(v^i) \to (T_xf)(v)\left(\left(\frac{\partial f^i}{\partial x^j}\right)_{f\circ\varphi^{-1}_{(x)}} v^j\right)$$

where $f^i = f^i(x^1, \dots, x^n)$, i=1, ... , n, is the expression of the application f in local coordinates.

2. EXAMPLES OF TANGENT VECTOR SPACES

Example 2.1 Let $GL(n, \mathbf{R}) = \{ A \in M_{n\times n}(\mathbf{R}) \mid \det(A) \neq 0 \}$. We will prove that

$$T_{I_n} GL(n, \mathbf{R}) = M_{n\times n}(\mathbf{R}).$$

Proof: We will show that $T_{I_n} GL(n, \mathbf{R}) \supseteq M_{n\times n}(\mathbf{R})$.

Let $A \in M_{n\times n}(\mathbf{R})$. We consider the curve:

$$c : \mathbf{R} \to GL(n, \mathbf{R})$$
$$c(t) = \exp(tA)$$

We have $c(0) = \exp(O_n) = I_n \Rightarrow \dot{c}(0) \in T_{I_n} GL(n, \mathbf{R}) \Rightarrow \frac{d\exp(tA)}{dt}\Big|_{t=0} \in T_{I_n} GL(n, \mathbf{R})$.

But

$$\frac{d\exp(tA)}{dt}\Big|_{t=0} = A e^0 = A.$$

It will result that $A \in T_{I_n} GL(n, \mathbf{R}) \Rightarrow T_{I_n} GL(n, \mathbf{R}) \supseteq M_{n\times n}(\mathbf{R})$.

Because $GL(n, \mathbf{R})$ is a differentiable manifold and

$$\dim M_{n\times n}(\mathbf{R}) = \dim T_{I_n} GL(n, \mathbf{R}) = n^2$$

we can conclude that

$$T_{I_n} GL(n, \mathbf{R}) = M_{n\times n}(\mathbf{R}).$$

Example 2.2 Let $(S^1, \mathbf{R}, A_{S^1})$ the unit circle with the differentiable structure given by:

$$S^1 = \{ (x^1, x^2) \in \mathbf{R}^2 \mid (x^1)^2 + (x^2)^2 = 1 \}$$
$$A_{S^1} = \{ h_N = (U_N, \varphi_N), h_S = (U_S, \varphi_S) \}$$

where:

$$U_N \overset{def}{=} S^1 - \{N = (0, 1)\} \text{ and } \varphi_N(x^1, x^2) \overset{def}{=} \frac{x^1}{1 - x^2}$$

$$U_S \overset{def}{=} S^1 - \{S = (0, -1)\} \text{ and } \varphi_S(x^1, x^2) \overset{def}{=} \frac{x^1}{1 + x^2}$$

We will determine now $T_N S^1$ and $T_S S^1$.

We consider the curve:

$$c : \mathbf{R} \to S^1$$
$$c(t) = (\sin t, \cos t).$$

We have:

$$c(0) = (\sin 0, \cos 0) = (0, 1) = N$$

It results that:

$$\frac{dc}{dt}\Big|_{t=0} = (\cos t, -\sin t) = (1, 0) \in T_N S^1.$$

Because $\dim T_N S^1 = 1 \Rightarrow T_N S^1 = \{ \alpha(1,0) \in T_N S^1 \}$.

For S = (0, -1) we will consider the curve:

$$c : \mathbf{R} \to S^1$$

$$c(t) = (\sin t, -\cos t).$$

We have:

$$c(0) = (\sin 0, -\cos 0) = (0, -1) = S$$

It results that:

$$\frac{dc}{dt}\Big|_{t=0} = (\cos t, \sin t) = (1, 0) \in T_N S^1.$$

Because $\dim T_S S^1 = 1 \Rightarrow T_S S^1 = \{ \alpha(1,0) \in T_S S^1 \}$.

We can conclude that $T_N S^1 = T_S S^1$.

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