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ON SOME PROPERTIES OF KANTOROVICH BIVARIATE OPERATORS

by
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Abstract. In this paper we continue our earlier investigations concerning the use of probabilistic methods for constructing linear positive operators useful in approximation theory of functions. The main result of this paper consists in introducing and investigating the approximation properties of Kantorovich bivariate operator, which is an integral linear positive operator reproducing the linear functions.

Keywords: bivariate operator

1. THE OPERATORS OF KANTOROVICH

Let $m \in N$ be fixed. The operators $K_m : L_1([0,1]) \to C([0,1])$, defined by

$$\left(K_m f\right)(x) = (m+1)\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt \tag{1.1}$$

are the operators of Kantorovich.

If we noticed χ_m the characteristic function of $\left(0, \frac{1}{m+1}\right]$ then the operators (1.1) can be define

$$\left(K_m f\right)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 f(t)\chi_m\left(t - \frac{k}{m+1}\right)dt, \tag{1.2}$$

where $p_{m,k} = \binom{m}{k}x^k(1-x)^{m-k}$, $k = \overline{0,m}$, $x \in [0,1]$ are the Bernstein polynomials.

Lemma[1]. The operators of Kantorovich satisfy the relations

i) $\left(K_m e_0\right)(x) = 1$,

ii) $(K_m e_1)(x) = \frac{m}{m+1}x + \frac{1}{2(m+1)}$,

iii) $(K_m e_2)(x) = \frac{m(m-1)}{(m+1)^2}x^2 + \frac{2m}{(m+1)^2}x + \frac{1}{3(m+1)^2}$.

iv) For all $f \in L_1([0,1])$, $(K_m f)(x) = \frac{d}{dx}(B_{m+1}F)(x)$, where $F(x) = \int_0^x f(t)dt$ and $(B_m f)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right)$, $x \in [0,1]$ are the operators of Bernstein.

Theorem [8]. The operators of Kantorovich have the properties

i) $\lim_{m\to\infty} K_m f = f$, uniformly on [0,1], $(\forall) f \in C[0,1]$.

ii) $\lim_{m\to\infty} K_m f = f$, $(\forall) f \in L_p[0,1]$, $p \geq 1$.

2. THE OPERATORS OF STANCU-KANTOROVICH

The Stancu polynomials [7] defined by

$$S_n^\alpha(f;x) = \sum_{k=0}^{n} w_{n,k}^\alpha(x) f\left(\frac{k}{n}\right) , \ x \in I = [0,1] , \ \alpha \geq 0$$

where $w_{n,k}^\alpha(x) = \binom{n}{k} \frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}$, $x^{(k,-\alpha)} = x(x+\alpha)...(x+(k-1)\alpha)$, can be used for constructing the operators of Stancu-Kantorovich[5]

$$K_n^\alpha(f;x) = (n+1)\sum_{k=0}^{n} w_{n,k}^\alpha(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \ . \qquad (2.1)$$

It is clear that the operators defined by (2.1) are linear and positive too. In the particular case $\alpha = 0$, the operator reduces obviously to the n^{th} classical Kantorovich operator defined by (1.1).

Lemma [3]. The operators defined by (2.1) satisfy the relations

i) $K_n^{\alpha}(1;x)=1$,

ii) $K_n^{\alpha}(t-x;x)=\dfrac{1-2x}{2(n+1)}$,

iii) $K_n^{\alpha}\left((t-x)^2;x\right)=x(1-x)\dfrac{n\dfrac{n\alpha+1}{\alpha+1}-1}{(n+1)^2}+\dfrac{1}{3(n+1)^2}$,

iv) If $0\le\alpha\le\dfrac{C}{n}$, with C a positive constant, it follows

$$K_n^{\alpha}\left((t-x)^2;x\right)\le C\left(\frac{1}{(n+1)^2}+\frac{x^2(1-x)^2}{n+1}\psi_n(x)\right)$$

where $\psi_n(x)=\begin{cases}1, & x\in E_n\\ 0, & x\in I\setminus E_n\end{cases}$, with $E_n=\left[\dfrac{A}{n},1-\dfrac{A}{n}\right]$ and A > 0 fixed.

Theorem. Let $\left(K_n^{(\alpha)}\right)_{n\ge1}$ be defined by (2.1) and

$$\beta(n,p)=\frac{\alpha(n)+(n+1)^{-1}}{\left(p+1^{1/p}\right)}+\frac{1}{3(n+1)^2} .$$

If $r\ge3$ is an integer number and $\beta^{1/r}(n,p)\le\dfrac{1}{2r}$, $n\in N$, the for every function $f\in L_p[0,1]$ we have

$$\left\|K_n^{(\alpha)}f-f\right\|\le C_{p,r}\left(\beta(n,p)\|f\|_p+\omega_r\left(2r\beta^{1/r}(n,p),f\right)_p\right)$$

with $C_{p,r}$ a positive constant independent of f and n.

3.THE BIVARIATE OPERATORS OF KANTOROVICH

We consider the operators defined by

$$(K_{mn}f)(x,y)=(m+1)(n+1)\sum_{k=0}^{m}\sum_{h=0}^{n}x^k(1-x)^{m-k}y^h(1-y)^{n-h}\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{h}{n+1}}^{\frac{h+1}{n+1}}f(u,v)dudv$$

where f belongs to the spaces $L_1([0,1]\times[0,1])$.

The operators defined by (3.1) are the bivariate operators of Kantorovich.

Lemma . The bivariate operators of Kntorovich satisfy the relations:

i) $(K_{mn}e_{00})(x,y)=1$,

ii) $(K_{mn}e_{10})(x,y)=\frac{m}{m+1}x+\frac{1}{2(m+1)}$,

iii) $(K_{mn}e_{01})(x,y)=\frac{n}{n+1}y+\frac{1}{2(n+1)}$,

iv) $(K_{mn}e_{11})(x,y)=\left(\frac{m}{m+1}x+\frac{1}{2(m+1)}\right)\left(\frac{n}{n+1}y+\frac{1}{2(n+1)}\right)$,

v) $(K_{mn}e_{20})(x,y)=\frac{m(m-1)}{(m+1)^2}x^2+\frac{2m}{(m+1)^2}x+\frac{1}{3(m+1)^2}$,

vi) $(K_{mn}e_{02})(x,y)=\frac{n(n-1)}{(n+1)^2}y^2+\frac{2n}{(n+1)^2}y+\frac{1}{3(n+1)^2}$.

Theorem . The bivariate operators of Kantorovich have the properties

i) $\lim_{n\to\infty}K_{mn}f=f$, uniformly on $[0,1]\times[0,1]$, $(\forall)f\in C([0,1]\times[0,1])$,

ii) $\lim_{n\to\infty}K_{mn}f=f$, $(\forall)f\in L_p([0,1]\times[0,1])$, $p\geq 1$.

4.THE BIVARIATE OPERATORS OF STANCU–KANTOROVICH

Now, we consider the operators defined by

$$K_{nm}^{\alpha}(f;x,y)=(n+1)(m+1)\sum_{k=0}^{n}\sum_{h=0}^{m}w_{n,k}^{\alpha}(x)w_{m,k}^{\alpha}(y)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{h}{m+1}}^{\frac{h+1}{m+1}}f(u,v)dudv \qquad (4.1)$$

where $w_{n,k}^{\alpha}(x)=\binom{n}{k}\frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}$, $x^{(k,-\alpha)}=x(x+\alpha)...(x+(k-1)\alpha)$ and $w_{m,h}^{\alpha}(x)=\binom{m}{h}\frac{y^{(h,-\alpha)}(1-y)^{(m-h,-\alpha)}}{1^{(m,-\alpha)}}$, $y^{(h,-\alpha)}=y(y+\alpha)...(y+(h-1)\alpha)$.

The operators defined by (4.1) are the bivariate operators of Stancu-Kantorovich.

Lemma . The operators defined by (4.1) satisfy the relations

i) $K_{nm}^{\alpha}(1;x,y)=1$,

ii) $K_{nm}^{\alpha}(u-x;x,y)=\frac{1-2x}{2(n+1)}$,

iii) $K_{nm}^{\alpha}(v-y;x,y)=\frac{1-2y}{2(m+1)}$,

iv) $K_{nm}^{\alpha}((u-x)(v-y);x,y)=\frac{1-2x}{2(n+1)}\cdot\frac{1-2y}{2(m+1)}$

v) $K_{nm}^{\alpha}\left((u-x)^2;x,y\right)=x(1-x)\frac{n\frac{n\alpha+1}{\alpha+1}-1}{(n+1)^2}+\frac{1}{3(n+1)^2}$,

vi) $K_{nm}^{\alpha}\left((v-y)^2;x,y\right)=y(1-y)\frac{m\frac{m\alpha+1}{\alpha+1}-1}{(m+1)^2}+\frac{1}{3(m+1)^2}$.

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