NECESSARY AND SUFFICIENT CONDITIONS FOR ALMOST REGULARITY OF UNIFORM BIRKHOFF INTERPOLATION SCHEMES

by
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Abstract: *In this article, using a combination of the necessary and sufficient conditions for the almost regularity of an interpolation scheme, we will determine all plane uniform Birkhoff schemes, when the set of interpolated nodes has* n *elements and the set of derivatives we are interpolating with is* $A = \{(0,0),(1,0)\}$.

For the same A *we will determine all rectangular Birkhoff uniform interpolation schemes (the set of nodes has rectangular shape) and we will take note of the fact that these two results differ significantly.*

Two criteria – normality condition and Pólya condition – are known to be used when establishing the almost regularity (regularity) of a multidimensional polynomial interpolation *general scheme.* Only one sufficient condition is known for the almost regularity (regularity) of a multidimensional interpolation scheme and this applies for a more restrictive domain, namely for the schemes of type Birkhoff.

Of course, for the interpolation schemes whose components are more and more limited, the number of the necessary and sufficient conditions for regularity (almost regularity) is increasing and these are more and more explicit. The two types of interpolation schemes that we will present in this article also prove this aspect.

We will present two necessary and sufficient conditions for bidimensional uniform Birkhoff schemes when the set that describes the derivatives is $A = \{(0,0),(1,0)\}$.

For the beginning we present the following notions:

1. The finite set $L \subset IN^2$ is *inferior* if $R(u,v) \subset L$ for any $(u,v) \in L$, where
$$R(u,v) = \{(i,j) \in IN^2 : i \le u, j \le v\}.$$

2. A set of nodes Z is (p,q)- *rectangular* or simple rectangular (p and q are natural numbers), if it can be written in the form
$$Z = \{(x_i, y_j) : 0 \le i \le p, 0 \le j \le q)\},$$
where $x_0, x_1, ...x_p$ are pair-wise distinctive real numbers (similarly for $y_0, y_1, ...y_q$).

3. The bivariate *uniform Birkhoff* interpolation scheme is the triplet (Z, S, A) consisting of a set (of nodes)

$$Z = \{ \mathbf{z}_t = (x_t, y_t) \in IR^2 \}_{t=1}^{n},$$

an inferior set $S \subset IN^2$ and a subset A of S. The associated (uniform, bivariate) Birkhoff interpolation problem consists in determining the polynomials

$$P \in P_S = \left\{ P \in IR[x,y] : P(\mathbf{z}) = \sum_{(i,j)\in S} a_{ij} x^i y^j, \mathbf{z} = (x,y) \in IR^2 \right\},$$

that satisfy the equations:

$$\frac{\partial^{\alpha+\beta} P}{\partial x^{\alpha} \partial y^{\beta}} (\mathbf{z}_t) = c_{\alpha,\beta}(\mathbf{z}_t), (\forall)(\alpha,\beta) \in A, \mathbf{z}_t \in Z,$$

where $c_{\alpha,\beta}(\mathbf{z}_t)$ are arbitrary real constants.

Moreover, if Z is rectangular then we have the *rectangular uniform Birkhoff* scheme.

4. An interpolation scheme (Z,S,A) is called *normal* if

$$|Z||A| = |S|,$$

(where $|Z|$, $|A|$ and $|S|$ is the cardinality of the corresponding sets etc.). In case of normality the determinant of the interpolated system exists and we denote it by $D(Z,S,A)$.

5. The scheme (Z,S,A) is *regular (singular)* if $D(Z,S,A)$ does not vanish (does vanish) for any choice of the set Z of nodes and is *almost regular* if $D(Z,S,A)$ is not identical null.

6. The interpolation scheme (Z,S,A) satisfies the *Pólya condition* if $|Z||A \cap L| \geq |L|$ for any inferior set $L \subset S$.

7. The conditions (criteria) *necessary* for the almost regularity of a general interpolation scheme (Z,S,A) are the normality condition and the Pólya condition.

8. A *sufficient* condition for the almost regularity of a uniform Birkhoff interpolation scheme is the following: if S admits a pavement (coverage) of unique type $\{A, A, ..., A\}$ (A n times), then (Z, A, S) is almost regular (a restatement of the results from [1]).

Next, we present the two promised necessary and sufficient conditions.

1. Proposition. *We consider the uniform Birkhoff scheme* (Z,S,A) *with* $A = \{(0,0),(1,0)\} \subseteq S$. *Then* (Z,S,A) *is almost regular with respect to the sets* Z

of n nodes if and only if the following conditions are satisfied:

(i) $|S| = 2n$ *and*

(ii) S *contains at most* n *elements on the axis* Oy.

Proof: The condition that must be satisfied at (i), when the scheme (S, A) is almost regular, results from the Pólya condition (6), applied to the inferior set $L = S \cap Oy$. For the reverse implication we use (8) and we will show that S admits a pavement of unique type with n copies of A. We use the induction after n. We consider the shift Λ_1, which moves (1,0) in (3,0) and the origin in (2,0). Thus Λ_1 is the shift relative to A which moves A "minimal" to the right. Then we consider the shift Λ_2 relative to $A \cup \Lambda_1(A)$ which moves A minimal to the right (thus it moves (1,0) in (5,0) and the origin in (4,0)). We continue in this mode, and when the first line of S is covered, we move to the next line that we fill in the same manner, i.e. from left to right. In this way we cover S with copies of A by the obvious "method" from left to right, line by line. If this process does not block up, this provides us a unique type pavement of S, with copies of A, and the proof is ended without even using the hypothesis of induction. On the other hand, the process can block up only because the points on the axis Oy can be covered moving the origin ($(1,0) \in A$, can bot be moved back toward left). In other words, when the process stops, the only points of S which have not been covered are on the axis Oy (and such points exist). Thus, in this case, if β is the biggest number with the property that $(0, \beta) \in S$, then $(1, \beta) \notin S$. In this case we start everything from the beginning. We consider the shift Λ, which moves A maximal upwards and then, maximal to the right (in S). Then $(0, \beta) \in \Lambda(A)$ and we can see that $S \setminus \Lambda(A)$ is a new inferior set which has $2(n-1)$ elements and which has one element less than S on the axis Oy, i.e. at most $n = (n-1)+1$ elements. Using the induction hypothesis, we find a pavement of the set $S \setminus \Lambda(A)$ with $n-1$ copies of A, and this fact, together with Λ gives us the desired pavement of the set S.

2. Proposition. *We consider the set of nodes* (p, q) *rectangular*

$$Z = \{(x_i, y_j) : 0 \le i \le p, 0 \le j \le q\}.$$

If S *is an inferior set, then* (Z, S, A) *is regular if and only if*

$$S = R(2p+1, q).$$

Moreover, in this case, the solutions $P \in P_S$ *of the interpolation equations*

$$\begin{cases} P(x_i, y_j) = c_{i,j} \\ \dfrac{\partial P}{\partial x}(x_i, y_j) = c'_{i,j} \end{cases}$$

(where $0 \le i \le p, 0 \le j \le q$ *, and* $c_{i,j}, c'_{i,j} \in IR$ *are arbitrary constants) will be given by*

$$P(x,y) = \sum_{i,j} c_{i,j}\varphi_i(x)\phi_j(y) + \sum_{i,j}[c'_{i,j} - \varphi'_i(x_i)c_{i,j}](x - x_i)\varphi_i(x)\phi_j(y),$$

where

$$\varphi_i(x) = \frac{(x-x_0)^2...(x-x_{i-1})^2(x-x_{i+1})^2....(x-x_p)^2}{(x_i-x_0)^2...(x_i-x_{i-1})^2(x_i-x_{i+1})^2....(x_i-x_p)^2},$$

$$\phi_j(y) = \frac{(y-y_0)...(y-y_{i-1})(y-y_{i+1})....(y-y_p)}{(y_i-y_0)...(y_i-y_{i-1})(y_i-y_{i+1})....(y_i-y_p)}.$$

Proof: We suppose first that (Z,S,A) is regular and we define

$$a_0 = \max\{a \in IN : (a,0) \in S\}, b_0 = \max\{b \in IN : (0,b) \in S\}.$$

First, it is clear that $S \subseteq R(a_0, b_0)$. Using the normality condition $|S| = |Z||A|$, we deduce that

$$2(p+1)(q+1) \le (a_0+1)(b_0+1). \tag{1}$$

On the other hand, we will show that

$$a_0 \le 2p+1. \tag{2}$$

We suppose the contrary, i.e. $a_0 \ge 2p+1$. Then a non null polynomial $P \in R[X]$ of degree a_0 exists such that

$$\begin{cases} P(x_i) = 0 \\ P'(x_i) = 0 \end{cases}$$

for any $0 \le i \le p, 0 \le j \le q\}$.

Indeed, this is a linear system (the unknowns are the coefficients of P) In which the number of the unknowns (a_0+1) is strictly bigger than the number of the equations ($2(p+1)$). Thus such P exists. It is clear that $P \in P_S$, and this contradicts the regularity of the scheme (Z,S,A).

It results analogously that

$$b_0 \le q. \tag{3}$$

Combining (1) (2) and (3), we see that the above inequalities, as well as the inclusion $S \subseteq R(a_0, b_0)$, become equalities. Thus $S = R(a_0, b_0)$, $a_0 = 2p+1$ and $b_0 = q$.

For the reciprocity is sufficient to check that the polynomial from the statement satisfies the interpolation conditions. Because of the symmetry, it is sufficient to check the equations in the node (x_0, y_0). We have:

$$\varphi_i(x_\alpha) = \begin{cases} 0, \ daca\ \alpha \neq i, \\ 1, \ daca\ \alpha = i, \end{cases}$$

$$\varphi'(x_\alpha) = 0 \text{, if } \alpha \neq i$$

$$\phi_j(y_\beta) = 0 \text{, if } \beta \neq j.$$

In particular, the polynomials $(x - x_i)\varphi_i(x)$ and $(x - x_i)\varphi'_i(x)$ vanish for any $x \in \{x_0, ..., x_p\}$. We deduce that, for any $0 \leq \alpha \leq p, 0 \leq \beta \leq q\}$,

$$P(x_\alpha, y_\beta) = \sum_{i,j} c_{i,j} \varphi_i(x_\alpha) \phi_j(y_\beta) = c_{\alpha,\beta} \ ,$$

$$\frac{\partial}{\partial x} P(x_\alpha, y_\beta) = \sum_{i,j} c_{i,j} \varphi'_i(x_\alpha) \phi_j(y_\beta) + \sum_{i,j} [c'_{i,j} - \varphi'_i(x_i) c_{i,j}\} \varphi_i(x_\alpha) \phi_j(y_\beta) =$$
$$= c_{\alpha,\beta} \varphi'_\alpha(x_\alpha) + (c'_{\alpha,\beta} - \varphi'_\alpha(x_\alpha) c_{\alpha,\beta}) = c'_{\alpha,\beta}.$$

q.e.d. □

3. Remark. The difference between the generic case and the rectangular case is clear. We consider for example

$$p = q = 1,$$

so we have $n = 4$ nodes. In the first case (the generic one) $s(Z, A) = 10$, and in the second case (the rectangular one) $s(Z, A) = 1$, where by $s(Z, A)$ we denoted the number of the inferior sets S for which (Z, S, A) is almost regular.

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