LOCALLY COMPACT BAER RINGS

by
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Abstract. Locally direct sums [W, Definition 3.15] appeared naturally in classification results for topological rings (see, e.g.,[K2], [S1], [S2], [S3], [U1]). We give here a result (Theorem 3) for locally compact Baer rings by using of locally direct sums.

1. Conventions and definitions

All topological rings are assumed associative and Hausdorff. The subring generated by a subset A of a ring R is denoted by $\langle A \rangle$. A *semisimple* ring means a ring semisimple in the sense of the Jacobson radical. A non-zero idempotent of a ring R is called *local* provided the subring eRe is local. The closure of a subset of a topological space X is denoted by $\overline{A}$. The Jacobson radical of a ring R is denoted by J(R). A *compact* element of a topological group [HR, Definition (9.9)] is an element which is contained in a compact subgroup. The symbol ω stands for the set of all natural numbers. All necessary facts concerning summable families of elements of topological Abelian groups can be found in [W, Chapter II,10, pp.71-80].

If R is a ring, $a \in R$, then $a^{\perp}=\{x \in R: ax=0\}$.

Recall that a ring R with identity is called a *Baer ring* if for each $a \in R$, there exists a central idempotent ε such that $a^{\perp}=R\varepsilon$.

The following properties of a Baer ring are known:

i) Any Baer ring does not contain non-zero nilpotent elements.

Indeed, let $a \in R$, $a^2=0$. Let $a^{\perp}=R\varepsilon$, where ε is a central idempotent of R. Then $a=a\varepsilon=0$.

ii) If R is a Baer ring, $a,b \in R$, n a positive natural number and $b^n a=0$, then $ba=0$.

Indeed, $b^{n-1}ab=0$, hence $b^{n-1}aba=0$. Continuing, we obtain that $(ba)^n=0$, hence $ba=0$.

Recall [K1,p.155] that a topological ring R is called a *Q-ring* provided the set of all quasiregular elements of R is open (equivalently, R has a neighbourhood of zero consisting of quasiregular elements).

Definition 1. A topological ring R is called *topologically strongly regular* if for each $x \in R$ there exists a central idempotent e such that $\overline{Rx} = Re$.

We note that a topologically strongly regular ring has no non-zero nilpotent elements.

Let $\{R_\alpha\}_{\alpha\in\Omega}$ be a family of topological rings, for each $\alpha\in\Omega$ let S_α be an open subring of R_α. Consider the Cartesian product $\prod_{\alpha\in\Omega} R_\alpha$ and let A=$\{\{x_\alpha\}\in\prod_{\alpha\in\Omega} R_\alpha$: $x_\alpha\in S_\alpha$ for all but finitely many $\alpha\in\Omega\}$. The neighborhoods of zero of $\prod_{\alpha\in\Omega} S_\alpha$ endowed with its product topology form a fundamental system of neighborhoods of zero for a ring topology on A. The ring A with this topology is called the *local direct sum* [W,Definition 31.5] of $\{R_\alpha\}_{\alpha\in\Omega}$ relative to $\{S_\alpha\}_{\alpha\in\Omega}$ and is denoted by $\prod_{\alpha\in\Omega}(R_\alpha : S_\alpha)$.

Definition 2. A topological ring R is called a *S-ring* if there exists a family $\{R_\alpha\}_{\alpha\in\Omega}$ of locally compact division rings with compact open subrings S_α with identity such that R is topologically isomorphic to the locally direct product $\prod_{\alpha\in\Omega}(R_\alpha : S_\alpha)$.

We will say that an element x of a topological ring R is *discrete* provided the subring Rx is discrete.

2. Results

Lema 1. Let $R_1,...,R_m$ be a finite set of division rings. If $\{e_\gamma : \gamma\in\Gamma\}$ is a family of non-zero orthogonal idempotents of $R=R_1\times\ldots\times R_m$ it is finite.

Proof. Assume the contrary, i.e., let there exists an infinite family $\{e_n : n\in\omega\}$of non-zero orthogonal idempotents. Then $\mathrm{Re}_0\subset \mathrm{Re}_0+\mathrm{Re}_1\subset\ldots$ is a strongly increasing chain of left ideals, a contradiction.

Theorem 2. A locally compact totally disconnected ring R is a S-ring if and only if it satisfies the following conditions:

i) R is topologically strongly regular,

ii) every closed maximal left ideal of R is a two-sided ideal and a topological direct summand as a two-sided ideal,

iii) every set of orthogonal idempotents of R is contained in a compact subring.

Proof. We note that if a locally compact totally disconnected ring satisfies the conditions i)-iii), then every its idempotent is compact.

(⇒) Let R=$\prod_{\alpha\in\Omega}(R_\alpha : S_\alpha)$, where each R_α is a locally compact totally disconnected ring with identity e_α and S_α is an open compact subring of R_α containing e_α.

i) Obviously.

ii) Let x=$\{x_\alpha\}\in$R. Denote $\Omega_0=\{\alpha\in\Omega : x_\alpha\neq 0\}$. Then $\varepsilon=\{\varepsilon_\alpha\}$, where ε_α=0 for $\alpha\notin\Omega_0$ and e_α otherwise, is a central idempotent of R. Obviously, x=xε, hence $\overline{Rx}\subseteq\overline{R\varepsilon}=R\varepsilon\subseteq R\varepsilon x\subseteq R\overline{x}$ and so $\overline{Rx}=R\varepsilon$.

iii) We claim that every closed maximal left ideal of R has the form $\{0_{\alpha_0}\}\times\prod_{\beta\neq\alpha_0}(R_\beta : S_\beta)$ for some $\alpha\in\Omega_0$. Indeed, every set of this form is a closed maximal left ideal of R.

Conversely, let I be a closed left ideal of R. Assume that $pr_\alpha(I)\neq 0$. for every α∈Ω. Then $pr_\alpha(I)=R_\alpha$ for every α∈Ω. There exists y=$e_\alpha\times\prod_{\delta\neq\alpha}x_\delta\in I$ and so e'_α =$e_\alpha\times\prod_{\beta\neq\alpha}0_\beta\in I$. For any x∈R, x∈$\overline{\langle e'_\alpha x : \alpha\in\Omega\rangle}\subseteq$I, a contradiction.

It follows that there exists $\alpha_0\in\Omega$ such that I$\subseteq\{0_{\alpha_0}\}\times\prod_{\beta\neq\alpha_0}(R_\beta : S_\beta)$. Since I is a maximal left ideal, I= $\{0_{\alpha_0}\}\times\prod_{\beta\neq\alpha_0}(R_\beta : S_\beta)$.

(⇐) Let now be ring R a totally disconnected locally compact ring satisfying i)-iii). Then R is is semisimple. Indeed, the Jacobson radical of R is closed [K2]. If 0≠ε∈J(R), then $\overline{Rx}=R\varepsilon$, ε is a central idempotent. Then 0≠ε∈J(R), a contradiction.

Then the intersection of all left maximal closed ideals will be equal to zero. It follows that any idempotent of R is central. Let I_0 is a closed left ideal of R. By assumption I_0 is a two-sided ideal and there exists an ideal R_0 such that R=$R_0\oplus I_0$ is a topological direct sum. Evidently, R_0 is a locally compact division ring; denote by e_0 the identity of R_0. Obviously, e_0 is a compact central idempotent of R.

Assume that we have constructed a family $\{e_\alpha : \alpha<\beta\}$ of orthogonal idempotents such that each Re_α is a locally compact division ring. By iii) the family $\{e_\alpha : \alpha<\beta\}$ lies in a compact subring, hence it is summable. Denote $\sum_{\alpha<\beta}e_\alpha$ =e and assume that R(1-e)≠0. Consider the Peirce decomposition R=Re⊕R(1-e). The ring R(1-e) satisfies the condition of Theorem. If R(1-e)=0, then e is the identity element of R. Assume that

R(1-e)≠0. Then there exists a non-zero idempotent $0\neq e_\beta\in R(1-e)$ such that $R(1-e)e_\beta=Re_\beta$ is a locally compact division ring.

This process may be continued and we obtain a family $\{e_\alpha: \alpha\in\Omega\}$ of orthogonal idempotents such that $1=\sum_{\alpha\in\Omega} e_\alpha$ is the identity of R and each R e_α is a division ring.

Fix a compact open subring W of R. We claim that R topologically isomorphic to $\prod_{\alpha\in\Omega}(Re_\alpha:We_\alpha)$. Indeed, put $\psi(r)=(r_\alpha)$ for each $r\in R$. Firstly we will prove that ψ is defined correctly. Let U be an open subring of R such that $rU\subseteq W$. There exists a finite subset $\Omega_0\subseteq\Omega$ such that $e_\alpha\in U$ for all $\alpha\notin\Omega_0$. Then for each $\alpha\notin\Omega_0$, $re_\alpha\in rU\subseteq W \Rightarrow re_\alpha\in W e_\alpha$.

It is easy to prove that ψ is an injective continuous ring homomorphism of R in $\prod_{\alpha\in\Omega}(Re_\alpha:We_\alpha)$.

ψ is dense in $\prod_{\alpha\in\Omega}(Re_\alpha:We_\alpha)$: It suffices to show that $\psi(R)\supseteq\oplus_{\alpha\in\Omega}Re_\alpha$. Indeed, if $r=r_{\alpha_1}+\ldots+r_{\alpha_n}\in R_{\alpha_1}+\ldots+R_{\alpha_n}$, then ψ(r)=r.

ψ is open on its image: Indeed, if U is a compact open subring of R then there exists a compact open subring U_1 of R such that $U_1W\subseteq U\cap W$. There exists a finite subset $\Omega_0=\{\alpha_1,\ldots,\alpha_n\}\subseteq\Omega$ such that $e_\alpha\in U_1$ for all $\alpha\notin\Omega_0$. Choose a compact open subring U_2 of R such that $U_2e_{\alpha i}\subseteq U$ for $i\in[1,n]$. Then $\psi(U)\supseteq U_2e_{\alpha_1}\times\ldots\times U_2e_{\alpha_n}\times\prod_{\alpha\neq\alpha_1,\ldots,\alpha_n}We_\alpha$: We claim that if $u_1,\ldots,u_n\in U_2, w_\alpha\in W_\alpha, \alpha\neq\alpha_1,\ldots,\alpha_n$, then the family $\{u_ie_i : i\in[1,n]\}\cup\{we_\alpha:\alpha\neq\alpha_1,\ldots,\alpha_n\}$ is summable. It suffices to show that the family $\{we_\alpha:\alpha\neq\alpha_1,\ldots,\alpha_n\}$ is summable in W. Let V be an arbitrary open ideal of W. There exists a finite subset $\Omega_1\subseteq\Omega$, $\Omega_1\supseteq\Omega_0$ such that $e_\alpha\in V$ for all $\alpha\notin\Omega_1$. Then for each $\alpha\notin\Omega_1$, $w_\alpha=w_\alpha e_\alpha\in WV\subseteq V$, therefore we have for each $\Omega_2\subseteq\Omega$, $\Omega_2\cap\Omega_1=\varnothing$, $\sum_{\beta\in\Omega_2}w_\beta\in V$. Therefore $\{we_\alpha:\alpha\neq\alpha_1,\ldots,\alpha_n\}$ is summable in R.

Denote $x=u_1e_{\alpha_1}+\ldots+u_ne_{\alpha_n}+\sum_{\alpha\neq\alpha_1,\ldots,\alpha_n}w_\alpha$. Then $x\in U$ and $\psi(x)=u_1e_{\alpha_1}\times\ldots\times u_ne_{\alpha_n}\times\prod_{\alpha\neq\alpha_1,\ldots,\alpha_n}w_\alpha$. We have proved that $\psi(U)\supseteq U_2e_{\alpha_1}\times\ldots\times U_2e_{\alpha_n}\times\prod_{\alpha\neq\alpha_1,\ldots,\alpha_n}We_\alpha$.

Theorem 3. Let R be a totally disconnected locally compact Baer ring. Then R is topologically isomorphic to a locally direct sum of locally compact Baer Q-rings which are locally isomorphic to locally compact Q-rings without nilpotent elements and without discrete elements.

Proof. Let V be a compact open subring of R. Each idempotent of R is central. There exists an idempotent $e\in R$ and a family $\{e_\alpha\}_{\alpha\in\Omega}$ of orthogonal local idempotents of V such that $e=\Sigma_{\alpha\in\Omega} e_\alpha$. Then R is topologically isomorphic to a direct topological product $\prod_{\alpha\in\Omega}(R_\alpha:Ve_\alpha))\times R(1-e)$.

Rings Re_α, $\alpha\in\Omega$, R(1-e) are local Q-rings. It suffices to show that every locally compact Baer Q-ring is locally isomorphic to a Q-ring without non-discrete elements.

Let V be an open compact quasiregular subring of R. We affirm that an element $x\in R$ is discrete if and only if xV=0. Indeed, if x is a discrete element then there exists a neighbourhood U of zero such that $Rx\cap U=0$. Choose a neighbourhood W of zero such that $Wx\subseteq V$. Then, evidently, Wx=0. There exists a natural number n such that $V^n\subseteq W$. Then $v^nx=0$ for each $v\in V$, hence xv=0. Then xV=0=Vx. (Actually we proved that in a topological ring without non-zero nilpotent elements the notion of a discrete element is symmetric.)

Denote by I the set of all discrete elements of R. Then I is an ideal of R. We affirm that $I\cap V=0$: if $x\in I\cap V$, then xV=0, hence $x^2=0$ which implies that x=0.

We affirm that R/I has no non-zero nilpotent elements: if $x^2\in I$, then $x^2V=0$. Then $x^2v=0$ for every $v\in V$, hence xv=0. We proved that xV=0, therefore $x\in I$.

We claim that R/I has no non-zero discrete elements. Let $x\in R, xW\subseteq I$ for some neighbourhood W of 0_R. Then xWV=0, hence $xV^n=0$ for some natural number n, hence xV=0, and so $x\in I$.

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