ABOUT SOME INEQUALITIES CONCERNING THE FRACTIONAL PART

by
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Abstract. The main purpose of this paper is to find the rational numbers x which have the property that $\{2^n \cdot x\} \geq \frac{1}{3}, \forall n \in \mathbf{N}$.

Key words: fractional part, length.

INTRODUCTION

If $n\in\mathbf{N}$, $n\geq 2$ has the standard decomposition $n = p_1^{a_1} \ldots p_r^{a_r}$, we define the length of n to be the number $\Omega(n) = \sum_{i=1}^{n} a_i$, $\Omega(1) = 0$. In [1] and [2] I showed that $\forall n\in\mathbf{N}$, $n > 3$, there exists the positive integers a, b such that $n = a + b$ and $\Omega(ab)$ is an even number. The second proof from [2] uses the following lemma: if $r\in\mathbf{N}^*$, $n\in\mathbf{N}$, and p_j has the usual meaning (the j-th prime number) and p is a prime number $p \equiv \pm 3$, there exist the natural numbers a_j, $j = \overline{1, r}$ such that

$$\{ p_1^{a_1} \ldots p_r^{a_r} \frac{n}{p} \} \leq \frac{1}{p_{r+1}} .$$

If $r = 1$, it results that there is an $a\in \mathbf{N}$ such that $\{2^a \cdot \frac{n}{p}\} \leq \frac{1}{3}$. Starting from this point I posed the problem of finding the rational numbers x such that $\{2^n \cdot x\} \geq \frac{1}{3}, \forall n\in\mathbf{N}$.

THE MAIN RESULTS

It is enough to consider the case when $x = \frac{n}{k}$ ($n, k \in \mathbf{N}$, $(n, k) = 1$) is a rational number $0 < x < 1$. After some multiplications with 2, we can suppose that k and n are odd numbers. If $1 > x > \frac{1}{2}$, then $\{2x\} = 2x - 1 < x$ and, after some multiplications with 2, then we can suppose that $\frac{1}{3} \le x < \frac{1}{2}$. We will prove now the main statement of the paper.

Proposition 1
Let x a rational number $x = \frac{n}{k}$, where n, k are coprime, odd natural numbers. The number x has the property that:

$$\frac{1}{3} \le x < \frac{1}{2}$$

and

$$\{2^m x\} \ge \frac{1}{3}, \forall m \in \mathbf{N}.$$

Then

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + \ldots + 2^{a_1} + 2^{a_0}}{2^{a_r+2} - 1}$$

where

$$a_0 = 0 < a_1 < a_2 < \ldots < a_r$$

are natural numbers which satisfy the inequalities

$$a_{i+1} - a_i \le 2,\ \forall i = \overline{0, r-1}\,.$$

$r \in \mathbf{N}$ and $a_r + 2$ is the smallest number $l \in \mathbf{N}^*$ for which

$$2^l \equiv 1.$$

Proof. We show by induction that $\forall m \in \mathbf{N}$ we have

$$[2^{m+2}x] = 2^{b_r} + 2^{b_{r-1}} + \ldots + 2^{b_1} + 2^{b_0}\,,$$

where $b_0 < b_1 < b_2 ... < b_r = m$ are natural numbers depending on m and satisfying the inequalities

$$b_{i+1} - b_i \leq 2, \ \forall \ i = \overline{0, r-1}.$$

b_0 could be only 0 or 1. For $m = 0$ the statement is obvious since $[4x] = 1$; the last equality holds since

$$\frac{1}{3} \leq x < \frac{1}{2}.$$

The same inequality shows that $[8x] = 2$ or $[8x] = 3 = 2 + 1$. This means that the statement is true for $m = 1$. Let us suppose that the statement is true for $m \in \mathbf{N}^*$ and we want to prove the statement for $m + 1$. Using the induction hypothesis we infer that

$$2^{m+2}x = [2^{m+2}x] + \{2^{m+2}x\} = 2^{b_r} + 2^{b_{r-1}} + ... + 2^{b_1} + 2^{b_0} \{2^{m+2}x\},$$

where $b_0 < b_1 < b_2 ... < b_r = m$ are natural numbers depending on m and satisfying the inequalities

$$b_{i+1} - b_i \leq 2, \ \forall i = \overline{0, r-1}.$$

b_0 could be only 0 or 1. We analyze first the case

$$\{2^{m+2}x\} < \tfrac{1}{2}.$$

We will show that in this case $b_0 = 0$. Let us suppose that $b_0 = 1$. It results that

$$2^{m+1}x = 2^{b_r - 1} + 2^{b_{r-1} - 1} + ... + 2^{b_1 - 1} + 2^{b_0 - 1} + \tfrac{1}{2}\{2^{m+2}x\}.$$

From this last equality we obtain (taking into account that $b_0 = 1$ and $\{2^{m+2}x\} < \tfrac{1}{2}$) that

$$\{2^{m+1}x\} < \tfrac{1}{2}\{2^{m+2}x\} < \tfrac{1}{4}.$$

The last inequality is impossible since from the hypothesis we know that

$$\{2^{m+1}x\} \geq \tfrac{1}{2}.$$

Therefore b_0=0. From the above equalities we obtain that

$$2^{m+3}x = 2^{b_r+1} + 2^{b_{r-1}+1} + ... + 2^{b_1+1} + 2^{b_0+1} + 2\{2^{m+2}x\},$$

which lead us (taking into account the fact that that $\{2^{m+2}x\} < \frac{1}{2}$ at the conclusion that

The properties of numbers b_i together with $b_0 = 0$ (then $b_0+1 = 1$) ensure us that the induction step is true in this case. We have to analyze the case

$$\{2^{m+2}x\} \geq \frac{1}{2}.$$

Using again the equality

$$2^{m+3}x = 2^{b_r+1} + 2^{b_{r-1}+1} + ... + 2^{b_1+1} + 2^{b_0+1} + 2\{2^{m+2}x\},$$

we obtain that

The properties of numbers b_i ensure us that also in this case the induction step is proved. We showed therefore by induction that $\forall m \in \mathbf{N}$ we have the identity

$$[2^{m+2}x] = 2^{b_r} + 2^{b_{r-1}} + ... + 2^{b_1} + 2^{b_0},$$

where $b_0 < b_1 < b_2 ... < b_r = m$ are natural numbers depending on m and satisfying the inequalities

$$b_{i+1} - b_i \leq 2, \ \forall i = \overline{0, r-1}.$$

b_0 could be only 0 or 1. Let $a_r + 2$ the smallest $l \in \mathbf{N}^*$ such that

$$2^l \equiv 1.$$

$a_r + 2$ exists since k is odd. We have $a_r + 2 \geq 2$ since $k \neq 1$(do not forget that $x = \frac{n}{k}$, n and k being coprime odd natural numbers; also we have $\frac{1}{2} \leq x < \frac{1}{2}$. Since $2^{a_r+2} \equiv 1$ it follows that

$$\{2^{a_r+2}\tfrac{n}{k}\} = \{\tfrac{n}{k}\} = \{x\} = x = 2^{a_r+2}x - [2^{a_r+2}x].$$

Taking into account these equalities and the statement proved above by induction, it results that

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + ... + 2^{a_1} + 2^{a_0}}{2^{a_r+2} - 1}$$

where

$$a_0 < a_1 < a_2 < ... < a_r$$

are natural numbers which satisfy the inequalities

$$a_{i+1} - a_i \leq 2,\ \forall i = \overline{0, r-1}.$$

a_0 is 0 or 1. We have to show that $a_0 = 0$. This result from

$$\{2^{a_r+2}x\} = x < \tfrac{1}{2}$$

and from the first case of the induction above.

We will show now that if

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + \ldots + 2^{a_1} + 2^{a_0}}{2^{a_r+2} - 1}$$

where

$$a_0 = 0 < a_1 < a_2 < \ldots < a_r$$

are natural numbers which satisfy the inequalities

$$a_{i+1} - a_i \leq 2,\ \ \forall i = \overline{0, r-1}\ \ (r \in \mathbf{N}),$$

then

$$\{2^m x\} \geq \tfrac{1}{2},\ \forall m \in \mathbf{N}.$$

For proving this statement it is enough to show that the number

$$y = \frac{2^{b_r} + 2^{b_{r-1}} + \ldots + 2^{b_1} + 2^{b_0}}{2^{b_r+2} - 1}$$

(where

$$b_0 = 0 < b_1 < b_2 < \ldots < b_s$$

are natural numbers which satisfy the inequalities

$$b_{i+1} - b_i \leq 2,\ \forall i = \overline{0, s-1};\ s \in \mathbf{N})$$

has the property that

$$\tfrac{1}{3} \leq y < \tfrac{1}{2}.$$

We have

$$y \leq \frac{2^{b_s} + 2^{b_s - 1} + \ldots + 2^0}{2^{b_s+2} - 1} = \frac{2^{b_s+1} - 1}{2^{b_s+2} - 1} < \tfrac{1}{2}.$$

For showing the second inequality we will consider two cases. The first one is when $b_s = 2l$; $l \in \mathbf{N}$. In this case we have

$$y \geq \frac{2^{2l} + 2^{2l-2} + ... + 2^2 + 1}{2^{2l+2} - 1} = \frac{1}{3}$$

If $b_s = 2l + 1$; $l \in$ **N** then $y \geq \frac{2^{2l+1} + 2^{2l-1} + ... + 2^1 + 1}{2^{2l+3} - 1} = \frac{2^{2l+3} + 1}{3(2^{2l+3} - 1)} > \frac{1}{3}$.

Using the same arguments as in the above Proposition we can show the following result:

Proposition 2

Let x a rational number, $x = \frac{n}{k}$, where n, k are coprime odd natural numbers. We suppose that x has the following property:

$$\frac{1}{5} \leq x < \frac{1}{4}$$

and

$$\{2^m x\} \geq \frac{1}{5}, \ \forall m \in \mathbf{N}.$$

Then

$$x = \frac{2^{a_r} + 2^{a_{r-1}} + ... + 2^{a_1} + 2^{a_0}}{2^{a_r+3} - 1}$$

where

$$a_0 = 0 < a_1 < a_2 < ... < a_r$$

are natural numbers which satisfy the inequalities

$$a_{i+1} - a_i \leq 3, \ \ \forall i = \overline{0, r} \ \ (r \in \mathbf{N}),$$

If there is an i ($0 \leq i \leq r$) such that

$$a_{i+1} - a_i = 3,$$

then

$$a_{i-1} = a_i - 1.$$

We denote

$$a_{r+1} = a_r + 3; \ a_{-1} = 0.$$

It will results that

$$a_{r-1} = a_r - 1,\ a_1 \leq 2.$$

The number $a_r + 3$ is the order of 2 in U($\mathbf{Z}_k$, ·).

Proof: The proof is similar with that of Proposition 1. The fact that $a_{r+1} = a_r + 3$ follows from the inequalities

$$\frac{1}{5} \leq x < \frac{1}{5}.$$

The only fact which has to be proved is that for any i ($0 \leq i \leq r$) such that

$$a_{i+1} - a_i = 3,$$

then

$$a_{i-1} = a_i - 1.$$

Replacing x by $\{2^{a_r - a_i} x\}$, we observe that it is enough to show the statement only for $i = r$. We have to show that $a_{r-1} = a_r - 1$. Let us suppose that

$$a_{r-1} \leq a_r - 2.$$

Then

$$x \leq \frac{2^{a_r} + 2^{a_r - 1} - 1}{2^{a_r + 3} - 1} < \frac{1}{5}.$$

The last inequality is equivalent with

$$2^{a_r+3} - 5 \cdot 2^{a_r} 2 - 5 \cdot 2^{a_r - 1} + 4 > 0$$

and

$$2^{a_r - 1} + 4 > 0.$$

The last inequality is obviously true since $a_r \geq 1$(if $a_r = 0$ then $x = \frac{1}{5} < \frac{1}{5}$; this is impossible). We obtained a contradiction since x is greater than $\frac{1}{5}$. The second part of the proof is identical with that of Proposition 1.

References

[1] A. Gica, The Proof of a Conjecture of Additive Number Theory, Journal of Number Theory 94, 2002, 80--89.

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