A CERTAIN CLASS OF QUADRATURES

by
Eugen Constantinescu

Abstract. Our aim is to investigate a quadrature of form:

$$\int_0^1 f(x)dx = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) + c_4 f(x_4) + c_5 f(x_5) + R(f) \qquad (1)$$

where $f:[0,1] \to IR$ is integrable, *R*(*f*) is the remainder-term and the distinct knots x_j an supposed to be symmetric distributed in [0,1]. Under the additional hypothesis that all x_j an of rational type (see(4)), we are interested to find maximum degree of exactness of such quadrature.

1 Introduction

Let $\prod_m$ be the linear space of all real polynomials of degree $\leq m$ and denote $e_j(t) = t^j,\ j \in N$. A quadrature of form

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} c_k f(x_k) + R(f) \qquad (2)$$

has degrees (of exactness) *m* if *R*(*h*) = 0 for any polynomial $h \in \prod_m$. If *R*(*h*) = 0 for all $h \in \prod_m$ and moreover $R(e_{m+1}) \neq 0$ it is said that (2) has the exact degree *m*. It is known that if (2) has degree *m*, then $m \leq 2n-1$. Likewise, there exists only one formula
(2) having maximum degree $2n-1$.

The aim of this paper is to study the formulas like (2) for *n* = 5 having some practical properties. Let us note that in this case, the optimal formula having maximum degree *m* = 9 is

$$\int_0^1 f(x)dx = \sum_{k=1}^{5} c_k f(x_k) + r(f)$$

$$x_k = \frac{1}{2} \pm \frac{1}{6}\sqrt{5 \pm 2\sqrt{\frac{10}{7}}}, 1 \leq k \leq 4, x_5 = \frac{1}{2}$$

It is clear that not all knots x_k are rational numbers.

Definition 1 Formula (1) is said to be of "*practical-type*", iff
i) the knots x_j are of form

$$x_1 = r_1, x_2 = r_2, x_3 = \frac{1}{2}, x_4 = 1 - r_2, x_5 = 1 - r_1 \qquad (4)$$

where r_1, r_2 distinct rational numbers from $\left[0, \frac{1}{2}\right)$.
ii) all coefficients c_1, c_2, c_3, c_4, c_5 are rational numbers with $c_1 = c_5$ and $c_2 = c_4$.
iii) (1) is of order *p*, with $p \geq 1$. Therefore, in case *n* = 5 a practical-type formula has the form

$$\int_0^1 f(x)dx = A\big(f(r_1) + f(1 - r_1)\big) + B\big(f(r_2) + f(1 - r_2)\big) + C \cdot f\left(\frac{1}{2}\right) + R(f) \qquad (5)$$

A, *B* being rational numbers, $C = 1 - 2(A + B)$, and when r_1, r_2 are distinct rational numbers from $\left[0, \frac{1}{2}\right)$.

Lemma 1 Let *s* be a natural number and suppose in (5) we have $R(h) = 0$ for all $h \in \prod_{2s}$ Then *R*(*g*) = 0 for every g from $\prod_{2s+1}$.

Proof. Let $H(x) = \left(x - \frac{1}{2}\right)^{2s+1}$. According to symmetry $\int_0^1 H(x)dx = 0$ and also $R(H) = 0$. Observe that $e_{2s+1}(x) \equiv x^{2s+1} = H(x) + h_1(x)$ with $h_1 \in \prod_{2s}$. Therefore $R(e_{2s+1}) = 0$ and supposing $g \in \prod_{2s+1}$ with $g(x) = a_0 x^{2s+1} + \ldots,$, we have $R(g) = a_0 \cdot R(e_{2s+1}) + R(h_2), h_2 \in R_{2s}$, that is *R*(*g*) = 0.

Lemma 2 If in (5) we have *R*(*h*) = 0 for every polynomial of degree $\cdot \leq 4$, then

(6)

$$A=\frac{10r_2^2-10r_2+1}{60(1-2r_1)^2(r_1-r_2)(1-r_1-r_2)}$$

$$B=\frac{10r_2^2-10r_1+1}{60(1-2r_2)^2(r_2-r_1)(1-r_1-r_2)}$$

$$C=\frac{8+40(r_1^2+r_2^2)-40(r_1-r_2)+240r_1r_2(1-r_1-r_2+r_1r_2)}{15(1-2r_1)^2(1-2r_2)^2}$$

Proof. We use standard method, namely by considering polynomials

$$l_j=\frac{\omega(x)}{(x-x_j)\omega'(x_j)},\ j\in\{1,2,3,4,5\},\ \omega(x)=\prod_{k=1}^{5}(x-x_k)$$

For instance, taking into account that

$$\omega'(x)=-\frac{1}{4}(1-2r_1)^2(r_1-r_2),\ \text{with } \delta=\frac{1}{2}$$

are finds

$$0=R(l_1)=\int_0^1 l_1(x)dx-Al_1(x_1)$$

and we conclude with

$$A=\frac{1}{\omega'(x_1)}\int_{-\frac{1}{2}}^{\frac{1}{2}} t[t-(1-2r_1)h][t^2-(1-2r_2)^2h^2]dt=$$

$$=\frac{10r_2^2-10r_2+1}{60(1-2r_1)^2(r_1-r_2)(1-r_1-r_2)}$$

In a similar way are finds coefficients *B* and *C*. Taking into account that (5) is symmetric, we give:

Corollary 1 Quadrature formula (5) has order, $m\geq 5$, if and only if the coeffcients are given by (6).

Lemma 3 If (5) has order $m, m \geq 6$, then r_1, r_2 must be distinct rational numbers from $(0,1]$ such that

$$560 r_1^2 r_2^2 + 56\left(r_1^2 + r_2^2\right) - 56\left(r_1 + r_2\right) + 560 r_1 r_2 \left(1 - r_1 - r_2\right) + 5 = 0 \qquad (7)$$

Proof. It is sufficient to impose condition $R(e_6) = 0, e_6(x) = x^6$. By considering $[a,b] = [-1,1]$, are find $R(e_6) = \frac{1}{7} - 2Ar_1^6 - 2Br_2^6 = 0$. Using Lemma 2, see (6) we obtains condition (7).

Corollary 2 Suppose that (5) is of practical-type. If r_1, r_2 are distinct rational numbers from $(0,1]$ such that equalities (6) and (7) are verified, then (5) has order *m* = 7.

Let us remark, that from above proposition implies that

$$r_1 + r_2 - 2r_1 r_2 \geq \frac{2}{7}$$

Corollary 3 The maximum order of *m* of practical-type quadratus formula at 5-knots satisfied $m \leq 7$.

Proof. Formulas like (7) having order *m* = 8 does not exist. the reason is that by assuming $m \geq 8$, then according to Lemma 1 we must have *m* = 9. But in this case numbers r_1 and r_2 are not rational (see (3)).

Lemma 4 Then does not exist pairs of rational numbers (r_1, r_2) which satisfy

$$560 r_1^2 r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560 r_1 r_2 (1 - r_1 - r_2) + 5 = 0.$$

Proof. The case $(1-2r_1)(1-2r_2) = 0$ is impossible. Further, consider

$$(1-2r_1)(1-2r_2) \neq 0$$

and let $1 - 2r_1 = \frac{p}{2}, 1 - 2r_2 = \frac{x}{y}, p, q, x, y \in Z, q > 0, y > 0$ with (*p*; *q*) = 1, (*x*; *y*) = 1.

Because $(1-2r_2)^2 = \frac{3\left[5 - 7(1-2r_1)^2\right]}{7\left[3 - 5(1-2r_1)^2\right]}$, we obtain $7x^2\left(3q^2 - 5p^2\right) = 3y^2\left(5q^2 - 7p^2\right)$. It follows that $x^2 \equiv 0 (\bmod 3)$ or $p^2 \equiv 0 (\bmod 3)$. Therefore *x* or *p* is divisible by 3, *x* = 0 (*mod* 3), *x* = 3*k* with $k \in Z$.

Then after dividing by 3, are finds $y^2(5q^2 - 7p^2) = 3 \cdot 7(3q^2 - 5p^2)$, with means that $5q^2 - 7p^2$ must be divisible by 3.
From (*x; y*) = 1 it is clear that *y* is not divisible by 3. Now

$$5q^2 - 7p^2 = 6(q^2 - p^2) - (q^2 + p^2) \equiv -(q^2 + p^2) \equiv 0 \ (mod\ 3)$$

implies $p^2 + q^2 \equiv 0(\text{mod}\,3)$ which is impossible unless $p \equiv q \equiv 0(\text{mod}\,3)$, which can't happen because (*p; g*) = 1.

Theorem 1 The practical quadratures at five knots, having maximal degree of exactness *m* = 5 are those of form

$$\int_0^1 f(x)dx = A[f(r_1) + f(1 - r_1)] + B[f(r_2) + f(1 - r_2)] + Cf\left(\frac{1}{2}\right) + R(f) \qquad (8)$$

where $R(f)$ is remainder, r_1, r_2 are distinct rational numbers from (0, 1] and

$$A = \frac{10r_2^2 - 10r_2 + 1}{60(1 - 2r_1)^2(r_2 - r_1)(1 - r_1 - r_2)}$$

$$B = \frac{10r_1^2 - 10r_1 + 1}{60(1 - 2r_2)^2(r_2 - r_1)(1 - r_1 - r_2)}$$

$$C = \frac{8 + 40(r_1^2 + r_2^2) - 40(r_1 - r_2) + 240r_1r_2(1 - r_1 - r_2 + r_1r_2)}{15(1 - 2r_1)^2(1 - 2r_2)^2}$$

Let us note that in quadrature formula from (8) we have

$$R(e_6) = \frac{560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5}{105} \cdot \frac{1}{2^6}$$

If by $[z_0, z_1, ..., z_k; f]$ is denoted the difference of a function $f : [0,1] \to IR$ at a system of distinct points $\{z_0, z_1, ..., z_k\} \subset [0,1]$, it may be shown that.

Theorem 2 Any partial quadratures at five knots, having maximal degree *m* = 5 may be written as

$$\int_0^1 f(x)dx = f\left(\frac{1}{2}\right) + \frac{1}{12}\left[r_1, \frac{1}{2}, 1-r_1; f\right] + \\ + \frac{3-5(1-2r_1)^2}{240} \cdot \left[r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right] + R(f) \tag{9}$$

where r_1, r_2 are distinct rational numbers from $(0,1]$.

2 Examples

In the following of $R_j(f), j \in N^*$, we shall denote the remainders terms in certain quadratures formulas.

Example 1. The closed formulas like (8) are obtained in case $r_2 = 1$, namely

$$\int_0^1 f(x)dx = A_0[f(0) + f(1)] + C_0 f\left(\frac{1}{2}\right) + B_0[f(r) + f(1-r)] + R_1(f) \tag{10}$$

where $r \in Q, r \in (0,1), R_1(e_6) = \frac{14(1-2r)^6 - 6}{105 \cdot 2^6}$ and

$$A_0 = \frac{1}{6} - \frac{1}{15(1-2r)^2}; \quad B_0 = \frac{1}{60r(1-2r)^2(1-r)}; \quad C_0 = \frac{3}{2} - \frac{2}{15(1-2r)^2}.$$

Example 2. For instance, when $(r_1, r_2) = \left(1; \frac{1}{2}\right)$, (10) gives

$$\int_0^1 f(x)dx = \frac{7}{90}[f(0) + f(1)] + \frac{16}{25}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + \frac{2}{25} f\left(\frac{1}{2}\right) + R_2(f)\right] \tag{11}$$

$R_2(e_6) = \frac{1}{21 \cdot 2^7}$.

Example 3. In case $(r_1, r_2) = \left(\frac{1}{2}; \frac{1}{4}\right)$ are finds

$$\int_0^1 f(x)dx = \frac{86}{45}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] - \frac{224}{45}\left[f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + \frac{107}{15} f\left(\frac{1}{2}\right) + R_3(f)\right] \tag{12}$$

$R_3(e_6) = \frac{115}{21 \cdot 2^{12}}$.

3 The remainder term
In order to investigated the remainder we use same methods as in [1]- [6].

Theorem 3 Let $m=\frac{1}{2}, h=\frac{1}{2}, x_1=r_1, x_2=r_2, x_3=\frac{1}{2}, x_4=1-r_2, x_5=1-r_1$.
If $\Omega(t)=\left[t^2-(1-2r_1)^2\cdot\frac{1}{4}\right]\left[t^2-(1-2r_2)^2\cdot\frac{1}{4}\right]$.

$$R(f)=\int_{-\frac{1}{2}}^{\frac{1}{2}} t^2\Omega(t)\left[\frac{1}{2}-t, r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1, \frac{1}{2}+t; f\right]dt.$$

Proof. Let $\omega(x)=\prod_{j=1}^{5}(x-x_j)$. Because our formula (8) is of interpolatory type, it follows that we have

$$\int_0^1 f(x)dx=\int_0^1 L_4(x_1,x_2,x_3,x_4,x_5;f)dx+R(f)$$

where $R(f)=\int_0^1 \omega(x)[x,x_1,x_2,x_3,x_4,x_5;f]dx$.

But $\int_0^1 f(1-x)dx=\int_0^1 f(x)dx$ and using the symmetry of knots $\{x_1,x_2,...,x_5\}$ we have

$$L_4\left(r_1,r_2,\frac{1}{2},1-r_2,1-r_1;f|1-x\right)=L_4\left(r_1,r_2,\frac{1}{2},1-r_2,1-r_1;f|x\right).$$

Further, the equality $\omega(1-x)=-\omega(x)$ gives

$$R(f)=-\int_0^1 \omega(x)\left[1-x,r_1,r_2,\frac{1}{2};1-r_2,1-r_1;f\right]dx$$

Therefore the remainder from (8) may be written as $R(f)=\frac{1}{2}\int_0^1 \omega(x)D(f;x)dx$ with

$$D(f;x) = \left[x, r_1, r_2, \frac{1}{2}; 1-r_2, 1-r_1; f\right] - \left[1-x; r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right] =$$
$$= 2\left(x-\frac{1}{2}\right)\left[x, r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right]$$

In this manner

$$R(f) = \int_0^1 \left(x-\frac{1}{2}\right)\omega(x)\left[x, r_1, r_2, \frac{1}{2}, 1-r_2, 1-r_1; f\right]dx$$

which is the same with (13).

Further for $g \in C[0,1]$ we use the uniform norm $\|g\| = \max_{x\in[a,b]}|g(x)|$.

Corollary 4 Let us denote

$$\omega(x) = (x-r_1)(x-r_2)(x-1+r_1)(x-1+r_2),\ J(r_1,r_2) = \int_0^1 \left(x-\frac{1}{2}\right)^2 |\omega(x)|dx$$

If $R(f)$ is the remainder in (8), then for $f \in C^6[0,1]$

$$|R(f)| \le \frac{1}{46080} J(r_1,r_2)\|f^{(6)}\|.$$

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Author:

Constantinescu Eugen
Department of Mathematics
"Lucian Blaga" University of Sibiu
Str. Dr. I. Rat¸iu, nr. 5-7
550012 Sibiu, Romˆania.
E-mail address: *egnconst68@yahoo.com*