CONSTRUCTION OF CUBATURE FORMULAS BY USING LINEAR POSITIVE OPERATORS

by
Ioana Taşcu,
Andrei Horvat-Marc

Abstract. We construct some cubature formulas for a hypercube using the polynomials approximation of Bernstein-Stancu type with the nodes $M_{i,j,k}^{\langle\alpha\rangle}$ having the coordinates

$$x_{m,i}^{\langle\alpha\rangle} = (i+\alpha)/(m+2\alpha),$$
$$y_{n,j}^{\langle\beta\rangle} = (j+\beta)/(n+2\beta),$$
$$z_{r,k}^{\langle\gamma\rangle} = (k+\gamma)/(r+2\gamma),$$

when α, β, γ are non negative real parameters.

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1. Introduction

In order to construct cubature formulas for a parallelepipedic domain, we shall use some classes of linear positive interpolating operators for functions of several variables.

We want to approximate a multiple integral of order s extended to a hyperparallelepiped $D_s = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_s, b_s]$ for a function $f : D_s \to \mathbf{R}$. It is known that by linear changes of variables, the domain D_s can be transformed into a hypercube $\Omega_s = [0,1]^s$. Hence, we construct some cubature formulas for a function $f : \Omega_s \to \mathbf{R}$. Such formulas have the following form:

$$\iint_{\Omega_s} w(x_1, x_2, \ldots, x_s)\, dx_1 dx_2 \ldots dx_s = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \ldots \sum_{i_s=0}^{m_s} A_{i_1, i_2, \ldots, i_s} f\left(x_{i_1}^{(1)}, x_{i_2}^{(2)}, \ldots, x_{i_s}^{(s)}\right) + R_{m_1, m_2, \ldots, m_s}(f), \tag{1.1}$$

where $w(x_1, x_2, ..., x_s)$ is a weight function, $w : \Omega_s \to \mathbf{R}_+$. The last term of this formula is the remainder, or the complementary term, of the cubature formula (1.1), the points $M_{i_1,i_2,...,i_s}\left(x_{i_1}^{(1)}, x_{i_2}^{(2)}, ..., x_{i_s}^{(s)}\right)$ are the nodes and $A_{i_1,i_2,...,i_s}$ are the coefficients of this formula.

We remark that the nodes and the corresponding coefficients of this formula do not depend to the function f for which we approximate the weighted integral by using the multiple sum of order s from the second member of (1.1).

Such cubature formulas can be constructs in the following ways:

1) by using the undetermined coefficients method such that the formula has a given exactness degree and taking into account the number of parameters;
2) integrating the interpolation formulas of Lagrange, Newton or Biermann type. Also, we can use the extensions of the interpolation formulas of Bernstein or Hermite-Fejér type, which assure the uniform convergence of these interpolation procedures, when $f \in C(\Omega_s)$.

It is known that for the s-dimensional Lagrange interpolation procedure the uniform convergence can't be assured whatever is the select grid of nodes.

2. Cubature formula of Bernstein type in the space of functions $C(\Omega_s)$

Let us consider the extension of the Bernstein interpolation formula to s variables

$$f(x_1, ..., x_s) = \left(B_{m_1,...,m_s} f\right)(x_1, ..., x_s) + \left(R_{m_1,...,m_s} f\right)(x_1, ..., x_s),$$

where the Bernstein interpolation polynomial has the following expression

$$\left(B_{m_1,...,m_s} f\right)(x_1, ..., x_s) = \sum_{k_1=0}^{m_1} ... \sum_{k_s=0}^{m_s} p_{m_1,k_1}(x_1)...p_{m_s,k_s}(x_s) f\left(\frac{k_1}{m_1}, ..., \frac{k_s}{m_s}\right), \tag{2.2}$$

and the basic interpolation polynomials are

$$p_{m_i,k_i}(x_i) = \binom{m_i}{k_i} x_i^{k_i} (1 - x_i)^{m_i - k_i}, \quad i = \overline{1, s}.$$

The remainder of the formula (2.1) can be expressed by the means of the divided differences of second order, see D. D. Stancu [3]. If we consider the weight function identically equal with 1 on Ω_s and we integrate the interpolation formula (2.1), we obtain a cubature formula of the following form

$$\int_{\Omega_s}\ldots\int f\left(x_1,\ldots,x_s\right)dx_1\ldots dx_s=\frac{1}{(m_1+1)\ldots(m_s+1)}\sum_{k_1=0}^{m_1}\ldots\sum_{k_s=0}^{m_s}f\left(\frac{k_1}{m_1},\ldots,\frac{k_s}{m_s}\right)+R_{m_1,\ldots,m_s}(f), \tag{2.4}$$

because

$$\int_0^1 p_{m_i,k_i}(x_i)dx_i=\binom{m_i}{k_i}\frac{\Gamma(k_i+1)\Gamma(m_i-k_i+1)}{\Gamma(m_i+2)}=\frac{1}{m_i+1},\quad i=\overline{1,s}. \tag{2.5}$$

According to the results contained in the papers [3], the remainder of the cubature formula (2.4) has the following expression:

$$R_{m_1,\ldots,m_s}(f)=-\frac{1}{12}\sum_{j_1=1}^{s}\frac{1}{m_{j_1}}f_{x_j^2}(\xi_1,\ldots,\xi_s)-\frac{1}{12^2}\sum_{j_1,j_2=1}^{s}\frac{1}{m_{j_1}m_{j_2}}f_{x_{j_1}^2x_{j_2}^2}(\xi_1,\ldots,\xi_s)-\ldots$$

$$-\frac{1}{12^s}\frac{1}{m_1\ldots m_s}f_{x_1^2\ldots x_s^2}(\xi_1,\ldots,\xi_s),$$

where the derivative points belong to the hypercube Ω_s.

3. Cubature formulas of Bernstein-Stancu type

By using the approximation polynomials of Bernstein-Stancu type

$$\left(B_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}f\right)(x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r}p_{m,i}(x)p_{n,j}(y)p_{r,k}(z)f\left(\frac{i+\alpha}{m+2\alpha},\frac{j+\beta}{n+2\beta},\frac{k+\gamma}{r+2\gamma}\right) \tag{3.1}$$

where α,β,γ are non negative real numbers and $p_{m,i}$, $p_{n,j}$, $p_{r,k}$ are the basic interpolation polynomials of Bernstein type, we can construct a cubature formula for the 3-dimensional cube $D_3=[0,1]^3$ which is given by

$$\iiint_{D_3}f(x,y,z)dx\,dy\,dz=$$

$$=\frac{1}{(m+1)(n+1)(r+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{r}f\left(\frac{i+\alpha}{m+2\alpha},\frac{j+\beta}{n+2\beta},\frac{k+\gamma}{r+2\gamma}\right)+R_{m,n,r}^{\langle\alpha,\beta,\gamma\rangle}(f).$$

This formula has the exactness degree $(1,1,1)$.

We remark that all coefficients are positive and equal. If we assume that $f\in C^{2,2,2}(D_3)$ then the remainder can be expressed by the partial derivatives of order $(2,2,2)$ as it can be shown by means of the Peano theorem. Because the Peano kernels have constant sign on D_3, we can use the mean value theorem of integral calculus and the remainder has a representation given by the partial derivatives at a point $(\xi,\eta,\zeta)\in D_3$.

When we construct an approximation polynomial of global degree m for a tetrahedron $\Delta_3=\{(x,y,z),x\geq 0,y\geq 0,z\geq 0,x+y+z\leq 1\}$, we obtain the following representation

$$(B_m^{\langle\alpha,\beta,\gamma,\delta\rangle}f)(x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{k=0}^{m-i-j}p_{m,i,j,k}(x,y,z)f\left(\frac{i+\alpha}{m+2\alpha},\frac{j+\beta}{m+2\beta},\frac{k+\gamma}{m+2\gamma}\right), \quad (3.3)$$

where

$$p_{m,i,j,k}(x,y,z)=\binom{m}{i}\binom{m-i}{j}\binom{m-i-j}{k}x^i y^j z^k(1-x-y-z)^{m-i-j-k},$$

and α,β,γ are non negative real numbers.

By the Dirichlet trivariate integral

$$B(p,q,r,s)=_{\Delta_3}u^{p-1}v^{q-1}w^{r-1}(1-u-v-w)^{s-1}du\,dv\,dw,$$

we can obtain the following cubature formula for Δ_3:

$$\iiint_{\Delta_3}f(x,y,z)dxdydz=\frac{1}{(m+1)(m+2)(m+3)}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{k=0}^{m-i-j}f(\frac{i+\alpha}{m+2\alpha},\frac{j+\beta}{m+2\beta},\frac{k+\gamma}{m+2\gamma})+R_{m,n,s}^{\langle\alpha,\beta,\gamma\rangle}(f),$$

which has the exactness degree $(1,1,1)$. The remainder can be represent by means of the partial derivatives of orders $(2,0,0)$, $(0,2,0)$, $(0,0,2)$, $(2,2,0)$, $(2,0,2)$, $(0,2,2)$, $(2,2,2)$ in a certain point (ξ,η,ζ) of the tetrahedron Δ_3.

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Author:

Ioana Taşcu - North University of Baia Mare, Romania
Andrei Horvat-Marc - North University of Baia Mare, Romania