A FEW EVALUATION CRITERIA OF OPTIMUM FINANCING WITH RANDOM COSTS AND BENEFITS

by
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Abstract. This paper introduces a few solutions linked to the optimization of financing projects, as well as a few evaluation criteria of optimum financing with random costs and benefits.
Keywords: optimum financing, cost, benefit

Expressing the problem

We shall consider a financing project F under economic analysis that must be done by a specialized financial society and we shall assume that:

- the cost of the financing is a random variable X discrete or continuous whose probabilistic distribution is known or determined within certain conditions,

$$X : \begin{pmatrix} x \\ p(x) \end{pmatrix}$$

in which $x \geq 0$ represents a possible value of the financing cost and p(x) represents the probability of having such a cost in the discrete case or the density of probability in the continuous case, in such a way that

$$\sum_{x} p(n) = 1 \text{ in the first case and}$$

$$\int_{0}^{\infty} p(x)dx = 1 \text{ in the second case.}$$

- annual benefits for the whole period of investment resulted by financing are also discreet or continuous random variables which we note Y_k, $1 \leq k \leq n$, the period of investment being of n years; on the one hand we have got individual distributions

$$Y_k : \begin{pmatrix} y_k \\ p(y_k) \end{pmatrix}, \quad 1 \le k \le n$$

corresponding to each annual benefit and on the other hand we have got the common distribution of annual benefits represented by the random vector Y = (Y_1, …, Y_n).

Admitting that annual benefits can be real arbitrary numbers we can, also, write:

$$\sum_{y_k} p(y_k) = 1, \qquad \text{in the discrete case and}$$

$$\int_{-\infty}^{\infty} p(y_k) dy_k, \qquad \text{in the continuous case, for } \forall\ Y_k$$

- if the value the investment has after n years of functioning is called residual value, we shall note it Z and we shall suppose that Z is also a continuous or discrete random variable,

$$Z : \begin{pmatrix} z \\ p(z) \end{pmatrix} \quad \text{with } \sum_z p(z) = 1 \text{, in the discrete case and}$$

$$\int_0^{\infty} p(z)dz = 1\text{, in the continuous case.}$$

So, to a financing project F we can associate a random vector U = (X, Y, Z) = (X, (Y_1, …, Y_n), Z), where X represents the financing cost and (Y, Z) the benefits obtained.

We shall suppose that X, Y and Z are discrete random variables and we shall note M(X) the average financing cost $M(X) = \sum_x xp(x)$, with $D^2(X)$ the dispersion of the financing cost, $D^2(x) = \sum_x x^2 p(x) - \left[\sum_x xp(x) \right]^2$, with $M(Y_k)$ the average benefit for year k, $M(Y_k) = \sum_{Y_k} y_k p(y_k)$, with $D^2(Y_k)$ the dispersion of the annual benefit for the year k,

$D^2(Y_k) = \sum_{y_k} y_k^2 p(y_k) - \left[\sum_{y_k} y_k p(y_k) \right]^2$, with M(Z) the average residual value, $M(Z) = \sum_z zp(z)$, and with $D^2(Z)$ the dispersion of the residual value, $D^2(Z) = \sum_z z^2 p(z) - \left[\sum_z zp(z) \right]^2$.

We shall also note:

- $C(X, Y_k)$ – co-variation between the financing cost and the benefits for year k,

$$C(X, Y_k) = M(XY_k) - M(X)M(Y_k)$$

- C(X, Z) – co-variation between the financing cost and the residual value

$$C(X,Z) = M(XZ) - M(X)M(Z)$$

- $C(Yj, Y_k)$ – co-variation between the benefits of years j and k

$$C(Yj, Y_k) = M(YjY_k) - M(Yj)M(Y_k).$$

We note p = 100i the average annual percentage of interests demanded for the considered financing type.

We shall suppose that the variable X can take m values, which is

$$X : \begin{pmatrix} x_1, & x_2, & \ldots, & x_m \\ p(x_1), & p(x_2), & \ldots, & p(x_m) \end{pmatrix}, \sum_{j=1}^{m} p(x_j) = 1,$$

the annual benefit Y_k ill have n_k values, which is

$$Y_k : \begin{pmatrix} y_k^1, & y_k^2, & \ldots, & y_k^{n_k} \\ p(y_k^1), & p(y_k^2), & \ldots, & p(y_k^{n_k}) \end{pmatrix}, \sum_{s_k=1}^{n} p(y_k^{s_k}) = 1,$$

the residual value Z can also take r values,

$$Z : \begin{pmatrix} z_1, & z_2, & \ldots, & z_r \\ p(z_1), & p(z_2), & \ldots, & p(z_r) \end{pmatrix}, \sum_{q=1}^{r} p(x_q) = 1,$$

Thus we can notice that taking into consideration the above mentioned conditions, there is a V number of variants of financing development, V = m · (n_1 · …. · n_n) · r = $= m \cdot r \cdot \prod_{k=1}^{n} n_k$ (1) and represents the number of values the vector U = (X, Y, Z) can take.

If $(x_j, (y_1^{s_s}, y_2^{s_2}, ..., y_n^{s_n}), z_q)$ (2) is one of these variants, to whom we add an actualized total gross benefit

$$\sum_{k=1}^{n} y_k^{s_k} (1+i)^{-k} + z_q (1+i)^{-n} \tag{3}$$

as well as an actualized total net benefit

$$-x_j + \sum_{k=1}^{n} y_k^{s_k} (1+i)^{-k} + z_q (1+i)^{-n} \tag{4}$$

calculated in the conditions in which the allocation of capital takes place integrally at a certain moment, beginning from which, annual benefits are cashed each year

Thus we can introduce the random variables total actualized gross benefit, k, and the total actualized net benefit, B, thus:

$$K : \begin{pmatrix} u_t \\ p(u_t) \end{pmatrix}, 1 \le t \le \frac{V}{m} \tag{5}$$

and

$$B : \begin{pmatrix} v_d \\ p(v_d) \end{pmatrix}, 1 \le d \le V \tag{6}$$

In relation (5) u_t is one of the values (3), bearing the probability $p(y_1^{s_1}, ..., y_n^{s_n}, z_q)$, and in relation (6) v_d is one of the values (4) with the probability $p(x_j, y_1^{s_1}, ..., y_n^{s_n}, z_q)$.

One can easily notice that

$$B = - X + K \tag{7}$$

We can calculate the total medium actualized benefit M(K) and the total medium actualized net benefit M(B) starting from the definition of the average value of a random variable. Starting from the definition of the dispersion of a random variable, we can also calculate the dispersion of the total actualized gross benefit D^2 (K) and that of the actualized net benefit D^2 (B). The result is:

$$M(K)=\sum_{k=1}^{n}M(Y_k)(1+i)^{-k}+M(Z)\cdot(1+i)^{-n} \tag{8}$$

where

$$\text{M(B) = -M(X) + M(K)} \tag{9}$$

Also,

$$D^2(B)=D^2(X)+\sum_{k=1}^{n}D^2(Y_k)(1+i)^{-2k}+D^2(Z)(1+i)^{-2n} \tag{10}$$

if X, (Y_k) and Z are probabilistically independent variables and

$$\begin{aligned} D^2(B)&=D^2(X)+\sum_{k=1}^{n}D^2(Y_k)(1+i)^{-2k}+D^2(Z)(1+i)^{-2n}+\\ &+2\sum_{k=1}^{n}C(X,Y_k)(1+i)^{-k}+2C(X,Z)(1+i)^{-n}+2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}C(Y_j,Y_k)(1+i)^{-j-k}+\\ &+2\sum_{k=1}^{n}C(Y_k,Z)(1+i)^{-k-n}\end{aligned} \tag{11}$$

if X, (Y_k) and Z are probabilistically dependent.

In the firs case, $D^2(K)=\sum_{k=1}^{n}D^2(Y_k)(1+i)^{-2k}+D^2(Z)(1+i)^{-2n}$ and in the second case

$$\begin{aligned} D^2(K)&=\sum_{k=1}^{n}D^2(Y_k)(1+i)^{-2k}+D^2(Z)(1+i)^{-2n}+\\ &+2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}C(Y_j,Y_k)(1+i)^{-j-k}+2\sum_{k=1}^{N}C(Y_k,Z)(1+i)^{-k-n}\end{aligned}. \tag{12}$$

Observation

Under the above mentioned conditions, the optimization problem regarding the financing with random costs and benefits consists of analyzing and deducing if a financing project can be accepted by itself or by comparison to similar projects. The objective function can be the total medium actualized benefit (gross or net) or the dispersion of the total actualized benefit.

Criteria of evaluating optimum financing

1. The criterion of the medium value

Definition 1. We shall say that in proportion with the total medium actualized net benefit and with the fixed scale $0 < a < b$, a financing project F:

- is *accepted* if $M(B(F)) > b$,
- is *further studied* if $a \leq M(B(F)) \leq b$,
- is *rejected* if $M(B(F)) < a$.

We can similarly consider the gross benefit K or the returned gross or net benefits (rentabilities) $\frac{K}{X}$ or $\frac{B}{X}$. (13)

Definition 2. We consider the financing projects $F_1, F_2, \ldots, F_m$. We may say that in proportion to the total medium actualized net benefit and to a scale $b > 0$ the financing project F_{j_0}, $1 \leq j_0 \leq m$, is *the best* if

$$M(B(F_{j_0}) = \max_{1 \leq j \leq m} \{M(B(F_j)), b\}$$

(14)

Relation (14) and the evaluation criterion it introduced can be similarly formulated both for the total actualized gross benefit and for revenue benefits (13), using other scale each time.

It is obvious that changing the medium indicator from relation (14) the optimum solution can change, too. On the grounds of this criterion we can make up a classification of financing projects extremely useful especially when funds are limited.

2. The dispersion criterion

Definition 3. We shall say that in proportion with the total medium actualized net benefit and with the fixed scale $0 < c < d$ a financing project F:

- is *accepted* if $D^2(B(F)) < c$,
- is *further studied* if $c \leq D^2(B(F)) \leq d$
- is *rejected* if $D^2(B(F)) > d$.

Definition 4 We consider the compatible financing projects F_1, F_2, …, F_m. We can say that in proportion with the dispersion of the total actualized net benefit and with a scale d > 0 the financing project F_{j_0}, $1 \le j_0 \le m$, is *the best* if:

$$D^2(B(F_{j_0})) = \min_{1 \le j \le m} \{D^2(B(F_j)), d\} \qquad (15)$$

The two definitions may be reformulated for the total actualized gross benefit or for the total revenue benefits, being able to obtain another optimum solution each time.

Observation. The scale suggested for the two criteria presented can severely choose an optimum solution, but can also lead to problems without solutions. Thorough the way they are defined they can often diminish the risk of unprofitable financing.

3. The criterion medium – dispersion or aversion for risk

Due to the concept of medium value, the two above mentioned criteria bear certain weaknesses, although in practice they are quite useful. That is why another way of using both the medium value and the dispersion in a combination that is much easier to be put into practice has been searched.

Let us consider a financing project F with a total actualized net benefit B(F) and $M(B(F))=m$, $D^2(B(F)) = \sigma^2$ (16)

Let us also consider the expression:

$$E(F, \mu) = M(B(F)) - \mu D(B(F)) = m - \mu \cdot \sigma, \quad \mu \in R \qquad (17)$$

We may observe that $E(F, \mu)$ may be regarded as a medium profit corrected or modified of the project F, having

$$E(F, \mu) > m \text{ if } \mu < 0 \text{ or}$$
$$E(F, \mu) \le m \text{ if } \mu \ge 0.$$

Here μ is called *aversion coefficient against risk* in the following sense:

1. if $\mu > 0$, which is $E(F, \mu) < m$, in this case the decision factor is prudent and has aversion against risk and the other way round;

2. if $\mu < 0$, which is $E(F, \mu) > m$, then the decision factor is not prudent and does not have aversion against risk and the other way round;

3. if $\mu = 0$, which is $E(F, \mu) = m$, then the decision factor is indifferent or neuter against risk and the other way round.

We can observe without difficulty that the expression $E(F, \mu)$, as function of parameter μ is strictly decreasing.

Let us take into consideration now two financing projects F_1 and F_2 for which we shall have:

$E(F_1, \mu) = m_1 - \mu \cdot \sigma_1$ and $E(F_2, \mu) = m_1 - \mu \cdot \sigma_2$

Definition 5. We can say that in proportion with the aversion coefficient against risk the financing projects F_1 and F_2 *are found in a relation of equilibrium* if the following relation exists:

$$E(F_1, \mu) = E(F_2, \mu) \tag{18}$$

Relation (18) is called *condition or relation of equilibrium* in proportion with the aversion coefficient against risk of the financing projects considered.

Definition 6 The solution for the equilibrium equation (18) is called *equilibrium solution* or *indifference threshold* against risk in comparing the financing projects F_1 and F_2 considered.

We can easily draw the conclusion that the solution of the equation (18) is

$$\mu_0 = \frac{m_1 - m_2}{\sigma_1 - \sigma_2} \quad , \quad \sigma_1 \neq \sigma_2 \tag{19}$$

and $E(F_1, \mu_0) = E(F_2, \mu_0)$.

If $\sigma_1 = \sigma_2 \Rightarrow$ the equilibrium equation becomes $m_1 = m_2$, the projects being thus compared only by the total medium actualized net benefit.

Definition 7 Given the financing projects F_1 and F_2, then, in proportion with the aversion coefficient against risk, we can say that:

- project F_1 is preferred instead of project F_2, if $E(F_1, \mu) > E(F_2, \mu)$
- projects F_1 and F_2 are equally preferred if $E(F_1, \mu) = E(F_2, \mu)$.

Observation. We can observe that

$$E(F_1, \mu) - E(F_2, \mu) = m_1 - m_2 - \mu\,(\sigma_1 - \sigma_2) = (\sigma_1 - \sigma_2)(\mu_0 - \mu) \tag{20}$$

and thus, comparing σ_1 with σ_2 and μ with μ_0 we can draw the following relations:

1. If $\sigma_1 < \sigma_2$ and $\mu < \mu_0 \Rightarrow E(F_1, \mu) < E(F_2, \mu)$, so, project F_2 is preferred.

2. If $\sigma_1 < \sigma_2$ and $\mu > \mu_0 \Rightarrow E(F_1, \mu) > E(F_2, \mu)$, so, project F_1 is preferred.

3. If $\sigma_1 > \sigma_2$ and $\mu < \mu_0 \Rightarrow E(F_1, \mu) > E(F_2, \mu)$, so, project F_1 is preferred.

4. If $\sigma_1 > \sigma_2$ and $\mu > \mu_0 \Rightarrow E(F_1, \mu) < E(F_2, \mu)$, so, project F_2 is preferred.

The difficulty of this parameter is linked to the values of μ, in other words, it is linked to the way in which the aversion against risk of the decisional factor is measured.

Definition 8 The equation line $E = m - \sigma \mu$ is called the *line of aversion against risk*. The half-planes $\mu > 0$ and $\mu < 0$ are called *half-planes of high*, respectively *low aversion against risk.*

Observation. Considering that more than two projects are going to be analysed the criterion is more difficult to be put into practice. The lines (d_k): $E_k = m_k - \mu \cdot \sigma_k$, $1 \leq k \leq n$, do not have the same intersection point and thus, they do not admit, generally, the same indifference threshold against risk and then the procedure is applied step by step to every two projects.

4. The criterion for bankruptcy risk and the chance of enrichment

Definition 9 We consider F a financing project and B = B(F) the random variable for the total actualized net benefit of project F. Then:

a) $P(B < 0)$, the probability of realizing a negative total actualized net benefit is called *bankruptcy risk*;

b) $P(B = 0)$, the probability of realizing a null total actualized net benefit is called *stagnation risk*;

c) $P(B > 0)$, the probability of realizing a positive total actualized net benefit is called *enrichment chance*;

In the case in which the benefit of a project is a discrete random variable the following relation occurs:

$$P(B < 0) + P(B = 0) + P(B > 0) \quad (21)$$

Definition 10 In the sense of the bankruptcy risk the financing project F_{j_0} is the best if it is the less risky, which means:

$$P(B(F_{j_0}) = \min_{1\le j\le m}\{P(B(F_j)\langle 0)\} \qquad (22)$$

Definition 11 In the sense of the enrichment chance, the project F_{j_0} is the best if:

$$P(B(F_{j_0})) = \max_{1\le j\le m}\{P(B(F_j))\rangle 0)\} \qquad (23)$$

Observation. The evaluation criteria (22) and (23) can be improved through the introduction of a maximum risk threshold r and of a minimum earning threshold q. Then we shall have:

$$P(B(F_{j_0})) = \min_{1\le j\le m}\{P(B(F_j)\langle 0), r\} \qquad (24)$$

in the first case and

$$P(B(F_{j_0})) = \max_{1\le j\le m}\{P(B(F_j))\rangle 0), q\} \qquad (25)$$

in the second case.

The procedure is very interesting and forces the enumeration of all the variants of project development in order to find out which of them lead to negative benefits and which to positive benefits.

Observation. At this very moment as well as in the majority of cases dealing with optimization problems we can draw the conclusion that applying all the above mentioned criteria for a certain case we do not necessarily obtain the same optimum solution.

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