PERFECT MORSE FUNCTIONS AND SOME APPLICATIONS

by
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Abstract. In this paper the perfect Morse functions on a compact manifold are studied. Some constructions of such functions on compact manifolds and some applications are also given.
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1. Introduction

Consider M^m a m-dimensional closed manifold, i.e. a smooth m-dimensional manifold which is compact and without boundary. Let $C^\infty(M)$ be the real algebra of all smooth real mappings defined on M. Recall that for a mapping $f \in C^\infty(M)$ a critical point $p \in C(f)$ is *non-degenerated* if the bilinear form $(d^2f)_p : T_p(M)\times T_p(M)\to R$ is non-degenerated, i.e. there exists a chart (U,φ) at the point p such that Hessian matrix $H(f_\varphi)(\varphi(p))=(\partial^2 f_\varphi / \partial x^i \partial x^j(\varphi(p)))_{1\le I,\ j\ \le\ m}$ is nonsingular, where $f_\varphi = f\circ\varphi^{-1}$ denotes the locally representation of f in the chart (U,φ).

If the critical set C(f) contains only non-degenerated critical points, then the mapping f is called a *Morse function* on manifold M. Denote by $\Omega(M)$ the set of all Morse functions defined on M. It is well-known that $\Omega(M)$ is dense in $C^\infty(M)$ in the so called Whitney topology, therefore $\Omega(M)$ is not empty. For $f\in\Omega(M)$ let us denote by $\mu_k(f)$ the number of all critical points of f having the Morse index equal k, where $0 \le k \le m$. If $\mu(f)$ denotes the cardinal number of C(f), then the following decomposition holds:

$$\mu(f) = \sum_{k=0}^{m} \mu_k(f) \tag{1.1}$$

The number defined by

$$\gamma(M) = \min\{\mu(f) : f\in\Omega(M)\} \tag{1.2}$$

is called the *Morse-Smale characteristic* of manifold M. The number $\gamma(M)$ is intensively studied in the papers [1] – [5], [7], [16] - [18]. For $m \ge 7$ it represents a simply homotopy invariant of manifold M (see [13]).

The main result of Morse Theory is the following: for a Morse function $f\in\Omega(M)$ there exists a finite CW – complex which is homotopic equivalent to M and having $\mu(f)$ cells. In this respect the Morse-Smale characteristic $\gamma(M)$ points out these homotopic equivalent to M CW-complexes which have minimal number of cells.

2. Some results on perfect Morse functions

Because M^m is a compact manifold it follows that M has the homotopy type of a finite CW-complex. Therefore the singular homology groups $H_k(M;Z)$, $k=\overline{0,m}$, are finitely generated, thus for $k\in Z$

$$H_k(M;Z) = (\underbrace{Z\oplus ...\oplus Z}_{\beta_k times})\oplus(Z_{n_{k1}}\oplus ...\oplus Z_{n_{kb(k)}}), \qquad (2.1)$$

where $\beta_k(M;Z) = \beta_k(M;Z)$ are the Betti numbers of M related to the group $(Z,+)$, i.e.

$$\beta_k(M;Z)= \mathrm{rank}H_k(M;Z).$$

Consider $H_k(M;F)$, $k=\overline{0,m}$, the singular homology groups with the coefficients in the field F and $\beta_k(M;F) = \mathrm{rank}\ H_k(M;F) = \dim_F H_k(M;F)$, $k=\overline{0,m}$, the Betti numbers related to F.

If $f\in\Omega(M)$ the following important relations hold:

$$\mu_k(f)\geq\beta_k(M;F),\ k=\overline{0,m}\ \textit{(weak Morse inequalities)}$$

$$\sum_{k=0}^{m}(-1)^k\mu_k(f)=\chi(M)\ \textit{(Euler formula).}$$

For the proof and some interesting applications we refer to the book of Palais, R. S. and Terng, Chun-lian [15, p. 213-222].

The Morse function $f\in\Omega(M)$ is called *F-perfect* if

$$\mu_k(f) = \beta_k(M;F),\ k=\overline{0,m} \qquad (2.2)$$

The Morse function $f\in\Omega(M)$ is *exact* (or *minimal*) if

$$\mu_k(f) = \gamma_k(M),\ k=\overline{0,m} \qquad (2.3)$$

where $\gamma_k(M) = \min\{\mu_k(f) : f\in\Omega(M)\}$. Here $\mu_k(f)$ denotes the number of the critical points of f having the Morse index k.

In the sequel we are interested in the following:

Problem: when the manifold M has F perfect Morse functions for some fields F? Taking into account the weak Morse inequalities and the definition of $\gamma_k(M)$ one obtains that for any Morse function $f\in\Omega(M)$ and for any field F the following relations hold: $\mu_k(f) \geq \gamma_k(M) \geq \beta_k(M;F)$, $k = \overline{0,m}$. From these relations it follows that any F-perfect Morse function on M is exact. The problem of finding a Morse function on M arises from the problem of constructing a cellular decomposition of M in a sum of minimal number of cells.

Theorem 2.1 *The manifold M has F-perfect Morse functions if and only if*

$$\gamma(M) = \beta(M;F) \tag{2.4}$$

where $\beta(M;F) = \sum_{k=0}^{m} \beta_k(M;F)$.

Proof. Let $f \in \Omega(M)$ be a fixed Morse function. Using the weak Morse inequalities it follows that $\mu(f) \geq \beta(M;F)$, and from the definition of the Morse-Smale characteristic (see 1.2)) one obtains $\gamma(M) \geq \beta(M;F)$.

On the other hand if f is a F-perfect Morse function on M it follows that $\mu(f) = \beta(M;F)$, that is $\gamma(M) \leq \beta(M;F)$ and the desired relation (2.4) is obtained.

Conversely, if $\gamma(M) = \beta(M;F)$ there exists a Morse function $f \in \Omega(M)$ such that $\gamma(M) = \mu(f)$. From (1.1) and the definition of $\beta(M;F)$ one obtains $\sum_{k=0}^{m} (\mu_k(f) - \beta_k(M;F)) = 0$. Taking into account the weak Morse inequalities it follows $\mu_k(f) = \beta_k(M;F)$, $k = \overline{0,m}$, i.e. f is a F-perfect Morse function on M.

Lemma 2.2 *The following relations hold*

$$H_k(M;Q) = H_k(M;Z)\otimes Q,\ k = \overline{0,m} \tag{2.5}$$

and consequently $\beta_k(M;Z) = \beta_k(M;Q)$, $k = \overline{0,m}$.

Proof. According to the well-known universal coefficients formula for homology it follows

$$H_k(M;Q) = (H_k(M;Z)\otimes Q)\oplus Tor(Q;H_{k-1}(M;Z)),\ k\in Z$$

where $Tor(Q;H_{k-1}(M;Z))$ represents the torsion product of the Abelian groups $(Q,+)$ and $H_{k-1}(M;Z)$. Because $(Q,+)$ is without torsion the desired results are obtained.

Let $p \geq 2$ be a prime number. Taking into account the relations (2.1) let us denote

$$\alpha_k^j(p) = \begin{cases} 1 & if \quad p | n_{kj} \\ 0 & otherwise \end{cases}, \text{ j} = \overline{0, b(k)} \tag{2.6}$$

Consider

$$\text{d(k,p)} = \sum_{j=0}^{b(k)} \alpha_k^j(p), \text{ k} = \overline{0, m} \tag{2.7}$$

The following result represents a necessary and sufficient condition in terms of γ(M), β(M;Z) and d(k,p) in order that the manifold M has Z_p-perfect Morse functions.

Theorem 2.3 *The manifold M has Z_p-perfect Morse functions if and only if the following equality holds*

$$\gamma(M) = \beta(M;Z) + 2\sum_{k=0}^{m-1} d(k,p) + d(m,p) \tag{2.8}$$

Proof. From the universal coefficients formula for homology it follows

$$H_k(M;Z_p) = (H_k(M;Z)\otimes Z_p)\oplus \mathrm{Tor}(Z_p;H_{k-1}(M;Z)),\ k \in Z,$$

where $\mathrm{Tor}(Z_p;H_{k-1}(M;Z))$ is the torsion product of the groups $(Z_p, +)$ and $H_{k-1}(M;Z)$. Using the following well-known relations $Z_r\otimes Z_s \cong Z_{(r,s)}$, $Z\otimes Z_r \cong Z_r$, $\mathrm{Tor}(Z_r, Z_s) \cong Z_{(r,s)}$, $\mathrm{Tor}(Z_r, Z_r) = \{0\}$, where (r,s) represents the greatest common divisor of the integers r, s, from (2.1) one obtains

$$H_k(M;Z_p) = (\underbrace{Z_p \oplus \ldots \oplus Z_p}_{\beta_k(M;Z)times}) \oplus \left(\bigoplus_{j=1}^{b(k)} Z_{(n_{kj},p)} \right) \oplus \left(\bigoplus_{v=1}^{b(k-1)} Z_{(n_{k-1,v},p)} \right).$$

Taking into account the definition of the numbers d(k,p) given in (2.7) from the above relation it follows that $\beta_k(M;Z_p) = \beta_k(M;Z) + d(k,p) + d(k-1,p)$, $k = \overline{0,m}$. The relation (2.8) becomes $\gamma(M) = \beta_k(M;Z_p)$, i.e. from Theorem 2.1 the desired result follows.

Corollary 2.4 *Let p, q ≥ 2 be two prime numbers. The manifold M has simultaneously Z_p and Z_q-perfect Morse functions if and only if the equality (2.8) holds and*

$$d(m,p) - d(m,q) = 2\sum_{k=0}^{m-1} (d(k,q) - d(k,p)). \tag{2.9}$$

Corollary 2.5 *Let M^m be a simply-connected compact manifold without boundary and $m \geq 6$. If the homology groups $H_k(M;Z)$, $k = \overline{0,m}$ are without torsion then M has Z_p-perfect Morse functions for any prime number $p \geq 2$.*

Proof. Under the above hypotheses the Morse-Smale characteristic is given by $\beta(M;Z) + 2\sum_{k=0}^{m-1} b(k) + b(m)$ (see [3, Theorem 2.3]). Because $H_k(M;Z)$ is without torsion , $k = \overline{0,m}$, one obtains $b(k) = 0$, $k = \overline{0,m}$, thus the condition (2.8) is satisfied for any prime number $p \geq 2$.

Theorem 2.6 *Let M^m be a simply-connected compact manifold without boundary with $m \geq 6$. Then M has Q-perfect Morse functions if and only if the groups $H_k(M;Z)$, $k = \overline{0,m}$, have no torsion.*

Proof. Taking into account the above mentioned result (see [3, Theorem 2.3]) and Lemma 2.2 it follows that $\gamma(M) = \beta(M;Q) + 2\sum_{k=0}^{m-1} b(k) + b(m)$. Using Theorem 2.1 one obtains that M has Q-perfect Morse functions if and only if $2\sum_{k=0}^{m-1} b(k) + b(m) = 0$, i.e. if and only if $b(k) = 0$, $k = \overline{0,m}$.

Remark. The results in Corollary 2.5 and Theorem 2.6 can be extended if we replace the condition that the manifold M is simply-connected with that $\pi_1(M) = Z\oplus\ldots\oplus Z$ (s times), where $s \geq 0$ is an arbitrary integer and $\pi_1(M)$ represents the fundamental group of M. In this case we use the result given in V. V. Sharko [19] and the explicit formula for the Morse-Smale characteristic obtained in [3, Theorem 3.1(ii)].

3. Applications

1. Let us consider $S^m = \{\lambda \in R^{m+1} : \|\lambda\| = 1\}$, $m \geq 1$, the m-dimensional sphere in the Euclidian space R^{m+1}. It is well-known that $\gamma(S^m) = 2$. On the other hand the integer homology of S^m is given by

$$H_k(S^m;Z) = \begin{cases} Z & \text{if} \quad k = 0 \quad \text{or} \quad k = m \\ \{0\} & \text{otherwise} \end{cases} \tag{3.1}$$

thus $\beta_0(S^m;Z) = \beta_m(S^m;Z) = 1$ and $\beta_k(S^m;Z) = 0$ for $1 \leq k \leq m-1$.

Taking into account Theorem 2.1 and Theorem 2.3 the following result holds.

Theorem 3.1 *(i) S^m has Q-perfect Morse functions.*
(ii) For any prime number $p \geq 2$, S^m has Z_p-perfect Morse functions.

2. Let PR^m be the real projective m-dimensional space. It is well-known that PR^m is a compact differentiable smooth manifold, without boundary, and the integer homology of PR^m is

$$H_k(PR^m;Z) = \begin{cases} Z & if \quad k = 0 \\ Z_2 \quad if \quad k \quad is \quad odd, & 0 \prec k \prec m \\ Z \quad if \quad k \quad is \quad odd, & k = m \\ \{0\} & otherwise \end{cases}$$

If m is even from (3.2) one obtains $\beta_0(PR^m;Z) = 1$ and $\beta_k(PR^m;Z) = 0$ for $k \geq 1$. In this case it is easy to see that $d(1,2) = d(3,2) = \ldots = d(m-1,2) = 1$ and $d(k,2) = 0$, otherwise.

If m is odd from (3.2) it follows that $\beta_0(PR^m;Z) = \beta_m(PR^m;Z) = 1$ and $\beta_k(PR^m;Z) = 0$ for $1 \leq k \leq m-1$. It is not difficult to note that $d(1,2) = d(3,2) = \ldots = d(m,2) = 1$ and $d(k,2) = 0$, otherwise.

For a prime number $p \geq 3$ one obtains $d(k,p) = 0$, $k = \overline{0,m}$.

It is known (see the paper of N. H. Kuiper [14]) that the Morse-Smale characteristic of PR^m is $\gamma(PR^m) = m+1$. Using the above numbers $d(k,2)$, $d(k,p)$ for $p \geq 3$, and Theorem 2.1, Theorem 2.3 we derive the following result.

Theorem 3.2 *(i) PR^m has not Q-perfect Morse functions.*
(ii) PR^m has Z_2-perfect Morse functions.
(iii) For any prime number $p \geq 3$, PR^m has not Z_p-perfect Morse functions.

Notice that a Z_2-perfect Morse function is given in the paper of N. H. Kuiper [14].

3. Let $f : M^m \to R^{m+k}$ be a smooth mapping and let $f_v : M \to R$ be the real mapping defined by $f_v(x) = \langle f(x), v \rangle$, i.e. the height function with respect to the vector $v \in S^{m+k-1}$.

Consider the set

$$\mathcal{H}_f = \{f_v : v \in S^{m+k-1}\} \tag{3.3}$$

The mapping $f : M \to R^{m+k}$ is *nondegenerated* if f_v is a Morse function, for almost all vectors $v \in S^{m+k-1}$. It is known that any immersion $f : M \to R^{m+k}$ is nondegenerated.

If $f : M \to R^{m+k}$ is an immersion, then consider the (m+k)-dimensional manifold

$$N^f = \{(x,v) \in M \times R^{m+k} : x \in C(f_v)\}$$

and the vector bundle of rank k

$$N^f \xrightarrow{\pi} M,$$

where the projection π is defined by $\pi(x,v) = x$.

Consider the (m+k-1)-dimensional manifold

$$N^{f,1} = \{(x,v) \in M \times S^{m+k-1} : x \in C(f_v)\}$$

and the Gauss mapping $\mathcal{N} : N^{f,1} \to S^{m+k-1}$, where $\mathcal{N}(x,v) = v$. Because the immersion f is nondegenerated it follows that we can define the functions $\mu_j(f)$, $\mu(f) : S^{m+k-1} \to Z$, j = 0, 1, …, m, where

$$\mu_j(f)(v) = \begin{cases} \mu_j(f_v) & if \quad f_v \in \Omega(M) \\ 0 & otherwise \end{cases}$$

and

$$\mu(f)(v) = \begin{cases} \mu(f_v) & if \quad f_v \in \Omega(M) \\ 0 & otherwise \end{cases}.$$

Because f is a nondegenerated mapping it follows that the functions $\mu_j(f)$, $\mu(f)$, j = 0, 1, …, m, are integrable densities on S^{m+k-1}. Denote by v_{m+k-1} the volume of S^{m+k-1} and by σ_{m+k-1} the canonical Riemannian structure on S^{m+k-1}.

The number

$$\tau_j(f) = \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \mu_j(f) d\sigma_{m+k-1} \tag{3.4}$$

is called *the curvature of index j* of immersion $f : M \to R^{m+k}$, where j is a fixed integer with $0 \le j \le m$.

The number $\tau(f) = \sum_{j=0}^{m} \tau_j(f)$ represents *the absolute total curvature of f.*

It is easy to see that the following relation is true

$$\tau(f) = \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \mu(f) d\sigma_{m+k-1} \qquad (3.5)$$

Theorem 3.3 (the curvature inequalities) *The following inequalities hold:*

$$\tau_j(f) \geq \gamma_j(M) \geq \beta_j(M;F),\ j = 0, 1, \ldots, m$$

$$\tau(f) \geq \gamma(M) \geq \sum_{j=0}^{m} \gamma_j(M) \geq \sum_{j=0}^{m} \beta_j(M;F)$$

where F is any field.

Note that if the immersion f has the property that for almost all $v \in S^{m+k-1}$ *the Morse function* f_v *is F-perfect, then we have equalities in the above relations.*

Theorem 3.4 (Gauss-Bonnet type formula) The following relation holds:

$$\sum_{j=0}^{m} (-1)^j \tau_j(f) = \chi(M) \qquad (3.6)$$

where $\chi(M)$ *is the Euler-Poincaré characteristic of M.*

Proof. Taking into account the Euler formula $\sum_{j=0}^{m} (-1)^j \mu_j(f) = \chi(M)$, it follows that

$$\sum_{j=0}^{m} (-1)^j \tau_j(f) = \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \left(\sum_{j=0}^{m} (-1)^j \mu_j(f) \right) d\sigma_{m+k-1} =$$

$$= \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \chi(M) d\sigma_{m+k-1} = \chi(M)$$

and we are done.

Consider j a fixed integer with $0 \leq j \leq m$. The Morse function $g \in \Omega(M)$ is *j-tight* if $\mu_i(g) = \gamma_i(M)$ for all $i \leq j$. The Morse function $g \in \Omega(M)$ is *tight* if $\mu(g) = \gamma(M)$.

The immersion $f : M \to R^{m+k}$ is *j-tight (tight)* if any height function $f_v \in H_v \cap \Omega(M)$ is j-tight (tight).

If $f_0 : M \to R^{m+k}$ is a tight immersion, then for any immersion $f : M \to R^{m+k}$ the following inequality holds:

$$\tau(f) \geq \tau(f_0) \qquad (3.7)$$

Indeed, for $v \in S^{m+k-1}$ with f_v, $f_{0v} \in \Omega(M)$ one obtains $\mu(f)(v) = \mu(f_v) \geq \mu(f_{0v}) = \mu(f_0)(v)$, i.e.

$$\tau(f) = \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \mu(f) d\sigma_{m+k-1} \geq$$

$$\geq \frac{1}{v_{m+k-1}} \int_{S^{m+k-1}} \mu(f_0) d\sigma_{m+k-1} = \tau(f_0)$$

4. Exact cellular decompositions are structures that globally encode the topology of arobot' free space while locally described the free space's geometry. These structures have been widely used for path planning between two points, but can be used for mapping and coverage of robot free spaces. It is possible to present and define exact cellular decompositions where critical points of Morse functions indicate the location of cell boundaries. Also, it is possible to derive a general framework for defining decompositions in terms of critical points and then give examples, each corresponding to a different task. We can solve by using this method a relaxed form of Laplace's equation to find "fair" Morse function with a user-controlled number and configuration of critical points.

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