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REMARKS ON EPIREFLECTIVE SUBCATEGORIES OF THE CATEGORY OF TOPOLOGICAL MODULES

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Abstract. We give in this paper a characterization of epireflective subcategories of the category of topological modules.

Denote *R-TopMod* the category of all topologically left R-modules over a fixed topological ring R with identity.

A subcategory A of *R-TopMod* is epireflective if and only if A is closed under taking of closed submodules and topological products.

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1. Introduction

All topological rings are assumed to be associative, Hausdorff and with identity. All topological modules are assumed to be unitary and Hausdorff. For a topological ring R with identity *R-TopMod* denotes the category of all topological left R-modules. Denote by GAT_2 the category of all topological Abelian Hausdorff groups.

In [1] is given a characterization of subcategories of the category of Abelian topological groups which are epireflective in this category.

Recall that a full subcategory B of a category A is called *reflective* ([2], p. 255) provided for each $X \in A$ there is a $r(X) \in B$ and a morphism $r_X : X \to r(X)$ such that for each $Y \in B$ and each morphism $\alpha : X \to Y$ there is a unique $\beta : r(X) \to Y$ such that the following diagram is commutative,

$$\begin{array}{ccc} X & r_X\longrightarrow & r(X) \\ \alpha \searrow & & \swarrow \beta \\ & Y & \end{array}$$

i.e., $\alpha = \beta \circ r_X$.

A reflective subcategory is called *epireflective* ([2], p. 255)provided for each $X \in A$, α is an epimorphism.

We note only that morphisms of *R-TopMod* are continuous homomorphisms and objects are topological left R-modules. Recall that topological R-modules M and N are called *isomorphic* provided there exists a continuous isomorphism $\alpha : M \to N$ such that α^{-1} is a continuous isomorphism, too.

2. Epireflective subcategories of the category of topological modules

Lemma 1.*Let $M, N \in$ R-TopMod. Consider $\alpha : M \to N$ be a surjective continuous homomorphism and $\sigma : N \to M$ a continuous homomorphism such that $\alpha \circ \sigma = 1_N$. Then M is the topological direct sum of subgroups $\sigma(N)$ and* $\ker \alpha$.

Proof.

Let $m \in M$, then $m = \sigma \circ \alpha(m) + m - \sigma \circ \alpha(m)$ and $\alpha(m - \sigma \circ \alpha(m))$ $= \alpha(m) - \alpha(m) = 0$. Therefore $M = \sigma(N) + \ker \alpha$. Let $n \in N$ and $\sigma(n) \in \ker \alpha$. Then $\alpha \circ \sigma(n) = 0 = n$, hence $\sigma(n) = 0$ and so $\sigma(N) \cap \ker \alpha = 0$ i.e., $M = \sigma(N) \oplus \ker \alpha$ is an algebraic direct sum.

Let V be a neighborhood of zero of M. Choose a symmetric neighborhood U of zero of M such that $U - \sigma \circ \alpha(U) \subseteq V$. Then for each $u \in U$, $u = \sigma \circ \alpha(u) + u - \sigma \circ \alpha(u) \in (V \cap \sigma(N)) \oplus (V \cap \ker \alpha)$. Therefore $M = \sigma(N) \oplus \ker \alpha$ is a topological direct sum.

Theorem 2. *A subcategory A of R-TopMod is epireflective if and only if A is closed under taking of closed submodules and topological products.*

Proof.

($\Rightarrow$) Firstly we will show that A is closed under taking of topological products. Let $M_i \in A$, $i \in I$ and put $M = \prod_{i\in I} M_i$. For each $i \in I$ there exists $\alpha_i : r(M) \to M_i$ such that the following diagram

$$\begin{array}{ccc} M & r_M\longrightarrow & r(M) \\ pr_i \searrow & & \swarrow \alpha_i \\ & M_i & \end{array}$$

is commutative.

Define $\alpha : r(M) \to M$, $\mathrm{pr}_i(\alpha(x)) = \alpha_i(x)$ for all $x \in r(M)$ and $i \in I$. Obviously, α is a continuous homomorphism.

Then $\alpha \circ r_M = 1_M$. Let $m = (m_j)$, hence $\alpha \circ r_M(m_j) = (m_j)$, i.e., $\alpha \circ r_M = 1_M$.

We affirm that $r_M \circ \alpha = 1_{r(M)}$. Indeed, since r_M is an epimorphism, $r_M(M)$ is dense in $\overline{r(M)}$. Let $m = (m_j) \in M$; then $r_M \circ \alpha \circ r_M(m_j) = r_M(m_j) = 1_{r(M)} \circ r_M(m_j)$. It follows from the continuity of $r_M \circ \alpha$ and $1_{r(M)}$ that $r_M \circ \alpha = 1_{r(M)}$.

Since $r_M \circ \alpha(r(M)) = r(M)$, r_M is surjective. If $m \in M$, $r_M(m) = 0$, then $\alpha \circ r_M(m) = 0 = m$. Since $\alpha \circ r_M(M) = M$, α is surjective. If $\alpha(x) = 0$, $x \in r(M)$, then $r_M \circ \alpha(x) = 0 = x$, hence α is injective. Therefore α is a continuous isomorphism and so M and $r(M)$ are topologically isomorphic, hence $M \in A$.

Now we will show that A is closed under taking of closed submodules. Let $N \subseteq M$ be a closed submodule of $M \in A$. There exists a continuous homomorphism $\alpha : r(N) \to M$ such that the following diagram

$$\begin{array}{ccc} N & \xrightarrow{r_N} & r(N) \\ i \searrow & & \swarrow \alpha \\ & M & \end{array}$$

is commutative, where $i(n) = n$ for each $n \in N$.

We obtained $\alpha : r(N) \to M$. But $\alpha \circ r(N) = \alpha\left(\overline{r_N(N)}\right) \subseteq \overline{\alpha \circ r_N(N)} = \overline{N} = N$. Therefore we have a continuous homomorphism $r_N : N \to r(N)$ such that $\alpha \circ r_N = 1_N$. Since $\alpha \circ r_N(N) = N$, α is surjective. By Lemma **??**, $r(N)$ is a topological direct sum of $r_N(N)$ and $\ker \alpha$, hence $r_N(N)$ is closed in $r(N)$, therefore $r_N(N) = r(N)$.

We affirm that r_N and α are continuous isomorphisms: if $r_N(N) = 0$, then $\alpha \circ r_N(n) = n = 0$. If $\alpha(x) = 0$, where $x \in r(N)$, then there exists $n \in N$ such that $x = r_N(n)$,hence $\alpha \circ r_N(n) = 0 = n$ or $x = 0$. Therefore N and $r(N)$ are topologically isomorphic and so $N \in A$.

($\Leftarrow$) Let A be a subcategory of *R-TopMod* closed under taking of closed submodules and topological products. We affirm that A is epireflective in *R-TopMod* and will construct the reflexion.

Let $M \in A$. Consider the set of all pairs (N, α), where $\alpha : M \to N$ is a continuous homomorphism of M in a dense submodule of N and $N \in A$. We will say that (N, α) and (N', α') have the same class provided there exists a topological isomorphism $\beta : N \to N'$ such that the diagram

$$\begin{array}{ccc} M & \alpha\longrightarrow & N \\ \alpha' \searrow & & \swarrow \beta \\ & N' & \end{array}$$

is commutative.

Then classes form a set and denote by $\{(N_\alpha, \varphi_\alpha) : \alpha \in \Omega\}$ the set of all representatives of all classes. Consider the module $N = \prod_{\alpha\in\Omega} N_\alpha$ and let $r_M : M \to N$, $r_M(m) = (\varphi_\alpha(m))$. Denote by $r(M)$ the closure of $\overline{r_M(M)}$.

Let $M' \in R\text{-}TopMod$ and $\beta : M \to M'$ a homomorphism. Then $\overline{\beta(M)} \in A$ and the pair $\left(\overline{\beta(M)}, \beta\right)$ is equivalent to some $(N_{\alpha_0}, \varphi_{\alpha_0})$. Therefore there exists a topological isomorphism $\gamma_1 : N_{\alpha_0} \to \overline{\beta(M)}$ such that the following diagram

$$\begin{array}{ccc} M & \varphi_{\alpha_0}\longrightarrow & N_{\alpha_0} \\ \beta \searrow & & \swarrow \gamma_1 \\ & \overline{\beta(M)} & \end{array}$$

is commutative.

But we define $\gamma : r(M) \to \overline{\beta(M)}$, $\gamma(x) = \gamma_1 \circ \mathrm{pr}_{\alpha_0}(x)$, $x \in r(M)$. Then for each $m \in M$, $\gamma \circ r_M(m) = \gamma(\varphi_\alpha(m)) = \gamma_1 \circ \mathrm{pr}_{\alpha_0}(\varphi_\alpha(m)) = \gamma_1 \circ \varphi_{\alpha_0}(m) = \beta(m)$, i.e. $\gamma \circ r_M = \beta$.

The uniqueness of γ: Assume that $\gamma' : r(M) \to M'$ such that $\gamma' \circ r_M = \beta$. Then for each $m \in M$, $\gamma' \circ r_M(m) = \beta(m)$ and $\gamma \circ r_M(m) = \beta(m)$. Therefore $\gamma'|_{r_M(M)} = \gamma|_{r_M(M)}$. Since $r_M(M)$ is dense in $r(M)$, we get that $\gamma' = \gamma$.

Remark 3. *Obviously, the intersection of any family of epireflective subcategories of R-TopMod is epireflective.*

Corollary 4. *A subcategory A of GAT_2 is epireflective if and only if A is closed under taking of closed subgroups and topological products.*

Example 5. The category of all compact groups is epireflective. The reflexion of a topological group G is Bohr compactification of G. A set of generators of this category is $T = R/Z$.

Example 6. The category of all Abelian linear compact groups is epireflective. A set of generators of this category is the class of finite groups and quasicyclic groups.

References

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