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On some mathematical models of the discrete and continuous dynamical systems with applications

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Abstract. The linear algebra offers an important support of calculus for a lot of various mathematical problems. The usual methods of decomposition of a matrix can be used in the solution of different practical problems which can be mathematical modeled. The solutions of the first two mathematical models presented here and named "the model of a forest" and "the air model" are based on all above mentioned linear algebra techniques.

The third model named "the model of the two tanks" is inspired from the chemical problems of substances change between two tanks of chemical substances. In this paper will be presented two mathematical models of solution according to the discrete and the continuous case, respectively, and based on the matrix calculus and ordinary differential equations.

The model no. 4 is named "the model of the decomposition of a radioactive material" and is inspired from practical archaeological problems. The one of the most used method in the researches of archaeology and history is based on the decomposition of the ^{14}C. This paper present two mathematical models used in solving the above mentioned problems based on the theory of the discrete and continuous dynamical systems. The models will be, evidently, available in the study of every decomposition of chemicals substances. The first part of this paper is dedicated to a set of preliminary and necessary elements of linear algebra.

Definition 1. *We say that $\lambda \in \mathbf{R}$ is an eigenvalue of the $d \times d$ matrix A if $\det(A - \lambda I) = 0$. The set of all eigenvalues of a square matrix A is called the spectrum and denoted by $\sigma(A)$.*

Each $d \times d$ matrix has exactly d values. All the eigenvalues of a symmetric matrix are real, all the eigenvalues of a skew-symmetric are pure imaginary and, in general, the eigenvalues of a real matrix are either real or form complex conjugate pairs.

Definition 2. *We say that an eigenvalue is of algebraic multiplicity* $r \geq 1$ *if it is a zero of the characteristic polynomial* $p(z) = \det(A - zI)$,*in other words, if*

$$p(\lambda) = \frac{dp(\lambda)}{dz} = \ldots = \frac{d^{r-1}p(\lambda)}{dz^{r-1}} = 0, \ \frac{d^r p(\lambda)}{dz^r} = 0.$$

An eigenvalue of algebraic multiplicity one is said to be distinct.

Definition 3. *If* $\lambda \in \sigma(A)$ *it follows that* $\dim \ker(A - \lambda I) \geq 1$,*therefore there are nonzero vectors in the eigenspace* $\ker(A - \lambda I)$.*Each such vector is called an eigenvector of* A, *corresponding to the eigenvalue* λ.*An alternative formulation is that* $v \in \mathbf{R}^n \setminus \{0\}$ *is an eigenvector of* A, *corresponding to* $\lambda \in \sigma(A)$, *if* $Av = \lambda v$.

The geometric multiplicity of λ is the dimension of its eigenspace and it is always true that

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}.$$

Definition 4. *If the geometric and algebraic multiplicity are equal of its eigenvalues, A is said to have a complete set of eigenvectors.*

Since different eigenspaces are linearly independent and the sum of algebraic multiplicities is always d, a matrix possessing a complete set of eigenvectors provides a basis of $\mathbf{R}^d$ formed from its eigenvectors specifically, the assembly of all bases of its eigenspaces.

Lemma 1. *If a* $d \times d$ *matrix* A *has a complete set of eigenvectors then it possesses the spectral factorization*

$$A = VDV^{-1}. \tag{1}$$

Here D *is a diagonal matrix and* $d_{l,l} = \lambda_l$, $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$, *the* l^{th} *column of the* $d \times d$ *matrix* V *is an eigenvector in the eigenspace of* λ_l *and the columns of* V *are selected so that* $\det V \neq 0$, *or, in other words, so that the columns form a basis of* $\mathbf{R}^d$.

It is easy to verify that for a matrix A having a spectral factorization

$$A = VDV^{-1}$$

we shall have

$$A^n = VD^nV^{-1}. \tag{2}$$

1.The model of a forest

A forest contain a set of trees of different sizes: littles, averages and bigs. From statistical dates it can be considered that about $\frac{7}{18}$ of the little trees become average trees and about $\frac{2}{9}$ of the average trees become big trees. Every five years about $\frac{8}{35}$ of the average trees and $\frac{1}{10}$ of the big trees are cut and an equal number to the total number of the cut trees of little trees are introduced in the forest.

Taking care of all above mentioned dates we want to create a discrete model of the evolution in time of the forest.

We denote by x_1^0, x_2^0 and x_3^0 the initial number of little, average and big trees and with x_1, x_2, x_3 the variables that describe the evolution in time of the little, average and big trees in the forest. After five years we shall have

$$x_1 = x_1^0 - \frac{7}{18}x_1^0 + \frac{1}{10}\left(x_3^0 + \frac{2}{9}x_2^0\right) + \frac{8}{35}\left(x_2^0 + \frac{7}{18}x_1^0 - \frac{2}{9}x_2^0\right)$$

$$x_2 = x_2^0 + \frac{7}{18}x_1^0 - \frac{2}{9}x_2^0 - \frac{8}{35}\left(x_2^0 + \frac{7}{18}x_1^0 - \frac{2}{9}x_2^0\right)$$

$$x_3 = x_3^0 + \frac{2}{9}x_2^0 - \frac{1}{10}\left(x_3^0 + \frac{2}{9}x_2^0\right)$$

or, making the calculus

$$x_1 = 0.7x_1^0 + 0.2x_2^0 + 0.1x_3^0$$

$$x_2 = 0.3x_1^0 + 0.6x_2^0$$

$$x_3 = 0.2x_2^0 + 0.9x_3^0$$

or, using a matrix representation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0 \\ 0 & 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix}.$$

For the matrix $A = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0 \\ 0 & 0.2 & 0.9 \end{pmatrix}$ the eigenvalues can be found solving the equation $\det(A - \lambda I_3) = 0$ or

$$-\lambda^3 + 2.2\lambda^2 - 1.53\lambda + 0.32 = 0 \Leftrightarrow 100\lambda^3 - 220\lambda^2 + 153\lambda - 33 = 0$$

with the eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{6+\sqrt{3}}{10}$, $\lambda_3 = \frac{6-\sqrt{3}}{10}$ and the corresponding eigenvectors

$$v_1 = \left(\frac{35}{4}, -\frac{7}{2}, 1\right), v_2 = \left(\frac{\sqrt{3}-1}{2}, \frac{3-\sqrt{3}}{2}, 1\right),$$

$$v_3 = \left(\frac{\sqrt{3}+1}{2}, \frac{-3-\sqrt{3}}{2}, 1\right).$$

Taking care of (1) we get the next decomposition for the matrix A:

$$A = \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{6-\sqrt{3}}{10} & 0 \\ 0 & 0 & \frac{6+\sqrt{3}}{10} \end{pmatrix} \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \tag{3}$$

that means we have

$$P = \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{6-\sqrt{3}}{10} & 0 \\ 0 & 0 & \frac{6+\sqrt{3}}{10} \end{pmatrix}. \tag{4}$$

The evolution of the system will be, taking care of (2), the next:

$$A^n = PD^nP^{-1}.$$

Making the calculus and taking care of (4) we get

$$A^n = \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{6-\sqrt{3}}{10}\right)^n & 0 \\ 0 & 0 & \left(\frac{6+\sqrt{3}}{10}\right)^n \end{pmatrix} \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \tag{5}$$

The evolution of the system will be, taking care of (5), the next:

$$\begin{pmatrix} x_1^n \\ x_2^n \\ x_3^n \end{pmatrix} = A^n \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix} \tag{6}$$

It means that after an interval of time of $5n$ years the number of little, average and big trees can be find from (6). For example, after 10 years, corresponding to the case $n=2$ we shall have:

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix} = \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{6-\sqrt{3}}{10}\right)^n & 0 \\ 0 & 0 & \left(\frac{6+\sqrt{3}}{10}\right) \end{pmatrix} \times$$

$$\times \begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{pmatrix}$$

and computing the invert of the matrix P, we get

$$\begin{pmatrix} \frac{35}{4} & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ -\frac{7}{2} & \frac{3-\sqrt{3}}{2} & \frac{-3-\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \frac{98+3\sqrt{3}}{2401} \begin{pmatrix} 3 & \frac{1}{2} & -\frac{\sqrt{3}}{4} \\ \frac{4-\sqrt{3}}{2} & \frac{34-\sqrt{3}}{4} & \frac{-63+\sqrt{3}}{8} \\ \frac{-10+\sqrt{3}}{2} & \frac{-36+\sqrt{3}}{4} & \frac{56-\sqrt{3}}{4} \end{pmatrix}.$$

2. The air model

From statistical dates we have that if a day is a sunny day then the next day will be also a sunny day with a probability of 0.8 and will be a rainy day with a probability of 0.2. Also we know from statistical dates that if a day is rainy then the next day will be a rainy day with a probability of 0.7 or a sunny day with a probability of 0.3. Taking care of all above mentioned we want to create a mathematical model of the evolution in time of the weather in time starting from some initial conditions. Let us consider $X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ the random vector who describes the initial atmospheric conditions and there, the string $X^1 = \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}$, $X^2 = \begin{pmatrix} x_2^1 \\ x_2^2 \end{pmatrix}, \ldots, X^n = \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix}$ of random variables who describe the evolution after one, two, and so on, after n days of the weather.

After a day we have

$$\begin{cases} x_1^1 = 0.8x_1^0 + 0.3x_2^0 \\ x_2^1 = 0.2x_1^0 + 0.7x_2^0 \end{cases} \Leftrightarrow \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}.$$

After two days we shall have

$$\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}^2 \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

and after n days we shall have

$$\begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}.$$

Denoting by $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$ we get for the matrix A the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0.5$ and the corresponding vectors $v_1 = \left(\frac{3}{2}, 1\right)$, $v_2 = (1, -1)$. Taking care of (1) we have the next decomposition for the matrix A:

$$A = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

that means $P = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$.

From (2) we have for the matrix A^n the next representation:

$$A^n = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}^n \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

or, making the calculus

$$\begin{aligned} A^n &= \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.5)^n \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.5)^n \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix} \end{aligned} \tag{7}$$

So, the evolution in time of the system after n days and with the initial value $\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ will be given by the next representation:

$$\begin{pmatrix} x_1^n \\ x_2^n \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (0.5)^n \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}. \tag{8}$$

For example, if we have an entire day just a sunny day then, after three days, using the air model, we shall have taking care of (8) and from the initial conditions $\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the next $\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.35 \end{pmatrix}$ that means we shall have a sunny day with a probability of 0.65 and a rainy day with a probability of 0.35.

3.The model of the two tanks

Two tank contain water with salt (NaCl) and the concentrations of salt are about of 0.001g/l in the first tank and 0.01g/l in the second tank. Every minute from the first tank pass to the second tank a quantity of 1l and from the second tank to the first tank a quantity of 2l as it can be seen in Figure 1.

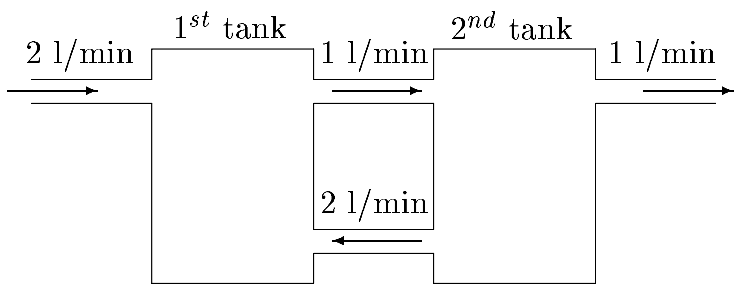

Figure 1:

Also, every minute a quantity of 2l of clear water come into the first tank and a quantity of 1l/min go out from the second tank.

Taking care of all these mentioned we want to express the evolution in time of the system that exactly means we want to know the evolution in time of the concentration of salt in the two tanks. We shall treat two cases: a discrete and a continuous one.

The discrete case

Let us consider:

V_1^k - the volume in the first tank after k minutes

N_1^k - the quantity of NaCl in the first tank after k minutes
V_2^k - the volume in the second tank after k minutes
N_2^k - the quantity of NaCl in the second tank after k minutes
having, evidently, the initial values:
$V_1^0 = V_2^0 = 100$ liters.

It is easy to see that the evolution of the volumes of the two tanks are given by:

$$\begin{aligned} V_1^{k+1} &= V_1^k + 3 \\ V_2^{k+1} &= V_2^k - 2 \end{aligned} \tag{9}$$

Also, taking care of the initial considerations, we get for N_1^k and N_2^k the next

$$N_1^{k+1} = N_1^k + \left(-\frac{N_1^k}{V_1^k} + 2\frac{N_2^k}{V_2^k} \right)$$

$$N_2^{k+1} = N_2^k + \left(\frac{N_1^k}{V_1^k} - 3\frac{N_2^k}{V_2^k} \right)$$

or

$$\begin{aligned} N_1^{k+1} &= N_1^k \left(1 - \tfrac{1}{V_1^k} \right) + 2\tfrac{N_2^k}{V_2^k} \\ N_2^{k+1} &= N_2^k \left(1 - \tfrac{3}{V_2^k} \right) + \tfrac{N_1^k}{V_1^k} \end{aligned} \tag{10}$$

We have to study the evolution of the concentrations of NaCl in the two tanks. For this, denoting by

$$p_1^k = \frac{N_1^k}{V_1^k}, p_2^k = \frac{N_2^k}{V_2^k}$$

we have, if we take care of (2), that

$$\begin{aligned} p_1^{k+1} &= \tfrac{N_1^{k+1}}{V_1^{k+1}} = p_1^k \tfrac{V_1^k - 1}{V_1^{k+1}} + 2p_2^k \tfrac{1}{V_2^{k+1}} \\ p_2^{k+1} &= \tfrac{N_2^{k+1}}{V_2^{k+1}} = p_1^k \tfrac{1}{V_2^{k+1}} + p_2^k \tfrac{V_2^k - 3}{V_2^{k+1}} \end{aligned} \tag{11}$$

From (1) we can deduce that

$$\begin{aligned} V_1^{k+1} &= 100 + 3(k+1) \\ V_2^{k+1} &= 100 - 2(k+1) \end{aligned} \tag{12}$$

and replacing (9) and (12) in (11) we get

$$p_1^{k+1} = \frac{99+3k}{103+3k} p_1^k + \frac{2}{103+3k} p_2^k$$

$$p_2^{k+1} = \frac{1}{98-2k} p_1^k + \frac{97}{98-2k} p_2^k$$

or

$$\begin{pmatrix} p_1^{k+1} \\ p_2^{k+1} \end{pmatrix} = \begin{pmatrix} \frac{99}{103+3k} & \frac{2}{103+3k} \\ \frac{1}{98-2k} & \frac{97}{98-2k} \end{pmatrix} \begin{pmatrix} p_1^k \\ p_2^k \end{pmatrix} \tag{13}$$

The continuous case

The evolution of the volumes in the tanks are given by:

$$\frac{dV_1}{dt} = +3(\mathrm{l/min})$$

$$\frac{dV_2}{dt} = -2(\mathrm{l/min})$$

and from here, taking care of the initial values in the tanks about 100 liters we get

$$\begin{array}{l} V_1(t) = 3t + 100 \\ V_2(t) = -2t + 100 \end{array} \quad \text{(in liters)} \tag{14}$$

The evolution of the NaCl quantities in the tanks:

$$\begin{aligned} \frac{dN_1}{dt} &= -\frac{N_1}{V_1} + 2\frac{N_2}{V_2} \\ \frac{dN_2}{dt} &= \frac{N_1}{V_1} - 3\frac{N_2}{V_2} \end{aligned} \tag{15}$$

The evolution of the concentrations of NaCl if we denote by $p_1 = N_1/V_1$, $p_2 = N_2/V_2$ will be given by

$$\begin{aligned} p_1' &= \frac{dp_1}{dt} = \frac{d(N_1/V_1)}{dt} = \frac{N_1'}{V_1} - \frac{N_1}{V_1^2} V_1' \\ p_2' &= \frac{dp_2}{dt} = \frac{d(N_2/V_2)}{dt} = \frac{N_2'}{V_2} - \frac{N_2}{V_2^2} V_2' \end{aligned} \tag{16}$$

Replacing (14) and (15) in (16) we get

$$\begin{aligned} p_1' &= \frac{1}{V_1}(-4p_1 + 2p_2) = \frac{1}{3t+100}(-4p_1 + 2p_2) \\ p_2' &= \frac{1}{V_2}(p_1 - p_2) = \frac{1}{-2t+100}(p_1 - p_2) \end{aligned} \tag{17}$$

which is a system of differential equations having the initial conditions:

$$p_1(0) = 0.001(1\text{gr/l})$$

$$p_2(0) = 0.01(10\text{gr/l}).$$

4. The model of the decomposition of a radioactive substance

It is well-known the fact that the time in which the ^{14}C gets in halves in every substance which contains it is about 5800 years. In other words denoting by $S = 5800$ years, after a period of S years the quantity of ^{14}C will be in halves, after a period of $2S$ years the quantity of ^{14}C will be in a measure of quarter and so on.

Supposing that we have initially a quantity of x_0 of ^{14}C after a period of S years we shall have a quantity

$$x_1 = \frac{1}{2}x_0, \tag{18}$$

after a period of $2S$ years we shall have a quantity

$$x_2 = \left(\frac{1}{2}\right)^2 x_0,$$

after periods of S years we shall have a quantity

$$x_R = \left(\frac{1}{2}\right)^R x_0.$$

Definition 5. *The system made by the string $X_0, X_1, \ldots, X_R, \ldots$ will be named as a discrete dynamical system.*

Like an example, let us consider that in an archaeological relic, the quantity of ^{14}C is found from measures about of 15% from the initial value. Taking care of these we want to find how old is the relic. From (20) we get

$$x_R = \left(\frac{1}{2}\right)^R x_0 = 0.15x_0 \Rightarrow \left(\frac{1}{2}\right)^R = 0.15 \Rightarrow R\ln\frac{1}{2} = \ln 0.15 \Rightarrow$$

$$R = \frac{\ln 0.15}{\ln 0.5} \approx 2.74(\text{periods of } S\text{years}).$$

The number of years of the relic is in this case $n \approx 2.74 \cdot S \approx 15800$ (years).

The continuous case

Let us consider that the quantity x of an radioactive material is a function of t, in other words $x = x(t)$. The rule of the decomposition will be given by:

$$-\frac{dx}{dt} = kx \Rightarrow \frac{dx}{dt} = -kx \Rightarrow \frac{dx}{x} = -kt \tag{21}$$

By integrating (4) we get

$$\int \frac{dx}{x} = \int -kdt \Rightarrow \ln x = -kt + C$$

or

$$x(t) = e^{-kt+C} = Ce^{-kt}$$

Denoting by $t_{1/2}$ the time in which the quantity of radioactive material get in halves we get

$$x(t_{1/2}) = \frac{x_0}{2} = x_0 e^{-kt_{1/2}} \Rightarrow \frac{1}{2} = e^{-kt_{1/2}} \Rightarrow$$

$$-kt_{1/2} = \ln \frac{1}{2} = -\ln 2 \Rightarrow k = \frac{\ln 2}{t_{1/2}}. \tag{23}$$

So

$$x(t) = x_0 e^{-\frac{\ln 2}{t_{1/2}} t} \tag{24}$$

or

$$x(t) = x_0 e^{\ln 2^{-t/t_{1/2}}} = x_0 2^{-t/t_{1/2}} = \frac{x_0}{2^{t/t_{1/2}}}$$

For $t = nt_{1/2}$ and denoting by $S = t_{1/2}$ we get

$$x(nS) = \frac{x_0}{2^n}$$

which is the same result with the result given by (20).

For example, about a radioactive material we know that after 300 years he had get at 10% of the initial value. We want to know $t_{1/2}$ and, also, the quantity of radioactive material who will remain after 400 years from the initial value.

We have:

$$x(300) = 0.1x_0$$
$$x(300) \overset{(7)}{=} x_0^{-\frac{\ln 2}{t_{1/2}}\cdot 300} \tag{25}$$

From (25) we deduce that

$$t_{1/2} = -\frac{\ln 2}{\ln 0.1} \cdot 300 \Rightarrow t_{1/2} \approx 90(\text{years}).$$

For $t = 400$ we get

$$x(400) \overset{(7)}{=} e^{-\frac{\ln 2}{t_{1/2}}\cdot 400} \approx e^{-\frac{\ln 2}{90}\cdot 400} \approx 0.046.$$

So, after 400 years, the quantity of radioactive material will be about 4.6% from the initial value.

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