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THE ASYMPTOTIC EQUIVALENCE OF THE DIFFERENTIAL EQUATIONS WITH MODIFIED ARGUMENT

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Abstract. This paper treats the asymptotic equivalence of the equations $x'(t) = A(t)x(t)$ and $x'(t) = A(t)x(t) + f(t, x(g(t)))$ using the notion of φ-contraction.

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1. Introduction

In 1964,W.A. Coppel [1] proposed an interesting application of Massera and Schäfer Theorem ([4],p. 530) obtaining the necessary and sufficient conditions for the existence of at least one solutions for the equations

$$x'(t) = A(t)x(t) + b(t) \tag{1}$$

for every $b(t)$ function.

More precisely, they consider $b \in C$, C being the class of the continuous and bounded functions defined on $\mathbf{R}_+ = [0, \infty)$ with the norm $\|b\| = \sup_{t \in \mathbf{R}_+} |b(t)|$, where $|\cdot|$ is the euclidian norm of $\mathbf{R}^n$.

W.A. Coppel([2],Ch.V) treated the case when $b \in L^1$, L^1 represents the Banach space of the Lebesgue integrable functions on $\mathbf{R}_+$with the norm $\|b\|_{L^1} = \int_{\mathbf{R}_+} |b(t)|dt$.

Using W.A.Coppel method in 1966 R.Conti [3] studied the same problem for the particular case when $b \in L^p$, $1 \le p \le \infty$, L^p being the space of the functions with $|b(t)|^p$ integrable on $\mathbf{R}_+$ with the norm $\|b\|_{L^p} = \left\{ \int_{\mathbf{R}+} |b(t)|^p dt \right\}^{\frac{1}{p}}$.

In 1968, Vasilios A. Staikos [6] studied the equation:

$$x' = A(t)x + f(t, x), \tag{2}$$

where the function f belongs to a class of functions defined on $\mathbf{R}_+$ and satisfies some restrictive conditions .

All along the mentioned paper the authors consider the subspace X_1 of the points in $\mathbf{R}^n$ which are the values of the bounded solutions for the equations

$$x' = A(t)x \tag{3}$$

at moment $t = 0$ and $X_2 \subseteq \mathbf{R}^n$is a supplementary subspace $\mathbf{R}^n = X_1 \bigoplus X_2$.

The fundamental conditions which was interpolated in W.A. Coppel paper, for equation (1) to have at least one bounded solution is the existence of projectors P_1 and P_2 and a constant $K > 0$ such that

$$\int_0^t |X(t)P_1X^{-1}(s)|ds + \int_t^\infty |X(t)P_2X^{-1}(s)|ds \leq K, \tag{4}$$

when $b \in C$,

$$\begin{cases} |X(t)P_1X^{-1}(s)| \leq K, & 0 \leq s \leq t \\ |X(t)P_2X^{-1}(s)| \leq K, & 0 \leq t \leq s \end{cases}, \tag{5}$$

when $b \in L^1$.

In their paper R. Conti and V.A. Staikos replaced conditions (4), and (5) with

$$\left(\int_0^t |X(t)P_1X^{-1}(s)|^p ds + \int_t^\infty |X(t)P_2X^{-1}(s)|^p\right)^{\frac{1}{p}} \leq K, \tag{6}$$

for $p \geq 1$ and

$$\sup_{0\leq s\leq t} |X(t)P_1X^{-1}(s)| + \sup_{t\leq s\leq\infty} |X(t)P_2X^{-1}(s)| \leq K, \tag{7}$$

for $p = \infty$.

In [6] Pavel Talpalaru consider the equation

$$x' = A(t)x \tag{8}$$

and the perturbed equation

$$y' = A(t)y + f(t, y), \tag{9}$$

where x, y, f are vectors in $\mathbf{R}^n$, $A(t) \in M_{n\times n}$,continuous in relation to t and y for $t \geq t_0$, $|y| < \infty$.

He demonstrated that under some conditions (see Theorem 2.1 from [7]) for all the bounded $x(t)$ solutions of the equation (8) there exists at least one $y(t)$ bounded solution (9), such that the next relation take place :

$$\lim_{t\to\infty} |x(t) - y(t)| = 0. \tag{10}$$

Next we introducing the notion of φ-contraction and comparison function by:

Definition 1.1.[8]$\varphi : R_+ \to R_+$ *is a strict comparison function if* φ *satisfies the following:*
i) φ *is continuous.*
ii)φ *is monotone increasing.*
iii) $\lim\limits_{n\to\infty} \varphi^n(t) \to 0$, *for all* $t > 0$.
iv) t-$\varphi(t) \to \infty$,*for* $t \to \infty$.

Let (X, d) be a metric space and $f : X \to X$ an operator.

Definition 1.2.[8] *The operator f is called a strict* φ*-contraction if:*
(i) φ *is a strict comparison function.*
(ii)$d(f(x), f(y)) \leq \varphi(d(x, y))$, *for all* $x, y \in X$.

In [8] I.A Rus give the following result:

Theorem 1.1.*Let* (X, d) *be an complete metrical space ,* $\varphi : R_+ \to R_+$ *a comparison function and* $f : X \to X$ *a* φ*-contraction.Then* f *, is Picard operator.*

Next we using the following lema:

Lemma 1.1.[6] *We suppose that X(t) is a continuous and invertible matrix for* $t \geq t_0$ *and let* P *an projector;If there exists a constant* $K > 0$ *such that*

$$\left\{ \int_{t_0}^{t} |X(t)PX^{-1}(s)|^q \right\}^{\frac{1}{q}} \leq K \ for \ \ t \geq t_0, \tag{11}$$

then there exists $N > 0$ *such that*

$$|X(t)P| \leq Nexp(-qK^{-1}t^{\frac{1}{q}}t^{1-\frac{1}{q}}) \ \ for \ \ t \geq t_0 \tag{12}$$

2. Main results

Let $t_0 \geq 0$. We consider the equation:

$$x'(t) = A(t)x(t), t \geq t_0 \tag{13}$$

and perturbed equation

$$y'(t) = A(t)y(t) + f(t, y(g(t))),\ t \geq t_0, \tag{14}$$

under conditions:
(a) $A \in M_{n\times n}$, continuous on $[t_0, \infty)$;
(b) $g : [t_0, \infty) \to [t_0, \infty)$, continuous;
(c) $f \in C([t_0, \infty) \times S)$, where $S = \{y \in \mathbf{R}^n \mid \ |y| < \infty\}$.

We note with C_α, the space of functions continuous and bounded defined on $[\alpha, \infty)$.

Theorem 2.1. *Let X(t) be a fundamental matrix of equation (13). We suppose that:*
(i) There exists the projectors P_1, P_2 and a constant $K > 0$ such that

$$\Big(\int_{t_0}^{t} |X(t)P_1X^{-1}(s)|^q ds + \int_{t}^{\infty} |X(t)P_2X^{-1}(s)|^q ds\Big)^{\frac{1}{q}} \leq K,$$

for $t \geq t_0,\ q > 1$;
(ii) There exists $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, comparison function , and $\lambda \in L^p([t_0, \infty)$ such that

$$|f(t, y) - f(t, y)| \leq \lambda(t)\varphi(|y - y|),$$

for all $t \geq t_0,\ y, y \in S$;
(iii) $f(\cdot, 0) \in L^p([t_0, \infty))$.

Then, for every solution bounded $x(t)$ of equation (13), there exists a unique solution bounded $y(t)$ of equation (14) such that

$$\lim_{t\to\infty} |x(t) - y(t)| = 0 \tag{15}$$

Proof. For $x \in C_{t_0}$ we consider the operator

$$Ty(t) = x(t)+\int_{t_0}^{t} |X(t)P_1X^{-1}(s)|f(s, y(g(s)))ds-\int_{t}^{\infty} |X(t)P_2X^{-1}(s)|f(s, y(g(s)))ds$$

We show that the space C_{t_0} is invariant for the operator T.If $y \in C_{t_0}$, then

$$|f(t, y(g(t))| \leq |f(t, y(g(t))) - f(t, 0)| + |f(t, 0)| \leq \lambda(t)\varphi(\|y\|) + |f(t, 0)|.$$

From:

$$\int_{t_0}^{\infty} |X(t)P_2X^{-1}(s)f(s, y(g(s)))|ds \leq$$

$$\leq \varphi(\|y\|)\Big(\int_{t_0}^{\infty} |X(t)P_2X^{-1}(s)|^q ds\Big)^1 q\Big(\int_{t_0}^{\infty} \lambda(s)^p ds\Big)^{\frac{1}{p}} +$$

$$+\Big(\int_{t_0}^{\infty} |X(t)P_2X^{-1}(s)|^q\Big)^{\frac{1}{q}}\Big(\int_{t_0}^{\infty} |f(s, 0)|^p\Big)^{\frac{1}{p}} \leq$$

$$\leq K\varphi(\|y\|)\Big[\Big(\int_{t_0}^{\infty} \lambda(s)^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t_0}^{\infty} |f(s, 0)|^p\Big)^{\frac{1}{p}}\Big]$$

we have that the definition of T is corect .

Let x a bonded solution for the equation (13) and $y \in C_{t_0}$. Then:

$$|Ty(t)| \leq |x(t)| + \int_{t_0}^{t} |X(t)P_1X^{-1}(s)f(s, y(g(s)))|ds +$$

$$+\int_{t}^{\infty} |X(t)P_2X^{-1}(s)f(s, y(g(s)))|ds \leq$$

$$\leq r+\int_{t_0}^{t} |X(t)P_1X^{-1}(s)|\cdot|f(s, y(g(s)))-f(s, 0|ds+\int_{t_0}^{t} |X(t)P_1X^{-1}(s)|\cdot|f(s, 0)|ds+$$

$$+\int_{t}^{\infty} |X(t)P_2X^{-1}(s)|\cdot|f(s, y(g(s)))-f(s, 0)|ds+\int_{t}^{\infty} |X(t)P_2X^{-1}(s)|\cdot|f(s, 0)|ds \leq$$

$$r + 2K\Big(\varphi(\|y\|)\Big(\int_{t_0}^{\infty} \lambda(s)^p ds\Big)^{\frac{1}{p}} + \Big(\int_{t_0}^{\infty} |f(s, 0)|^p\Big)^{\frac{1}{p}}\Big) < \infty$$

We show that the operator T is φ-contraction .

$$|Ty(t) - T\overline{y}(t)| \leq \int_{t_0}^{t} |X(t)P_1X^{-1}(s)| \cdot |f(s, y(g(s))) - f(s, \overline{y}(g(s)))|ds+$$

$$+\int_t^\infty |X(t)P_2X^{-1}(s)|\cdot|f(s,y(g(s)))-f(s,\overline{y}(g(s)))|ds\le$$

$$\le 2K\Big(\int_{t_0}^\infty \lambda(s)^p ds\Big)^{\frac{1}{p}}\varphi(\|y-\overline{y}\|)$$

We choose t_0 such that $\int_{t_0}^\infty \lambda(s)^p ds \le \frac{1}{2K}$.

From Theorem 1.1 we obtain that there exists a unique solutions of equation (14).

Let $y(t)$ be solution of (14) corespondent to $x(t)$.Then

$$|x(t)-y(t)|\le$$

$$\le \int_{t_0}^t |X(t)P_1X^{-1}(s)f(s,y(g(s)))|ds+\int_t^\infty |X(t)P_2X^{-1}(s)f(s,y(g(s)))|ds=I_1+I_2.$$

$$I_1=\int_{t_0}^t |X(t)P_1X^{-1}(s)f(s,y(g(s)))|ds$$

$$\le \int_{t_0}^{t_1} |X(t)P_1X^{-1}(s)f(s,y(g(s)))|ds+\int_{t_1}^t |X(t)P_1X^{-1}(s)f(s,y(g(s)))|ds\le$$

$$\le |X(t)P_1|\int_{t_0}^{t_1}|X^{-1}(s)||f(s,y(g(s)))|ds+K\varphi(\|y\|)\Big(\int_{t_1}^\infty \lambda(s)^p\Big)^{\frac{1}{p}}+K\Big(\int_{t_1}^\infty |f(s,0)|^p\Big)^{\frac{1}{p}}$$

We choice $t_1\ge t_0$ such that $\left(\int_{t_1}^\infty \lambda(s)^p\right)^{\frac{1}{p}}\le \frac{\varepsilon}{3K\varphi(\|y\|)}$, and $\left(\int_{t_1}^\infty |f(s,0)|^p\right)^{\frac{1}{p}}\le \frac{\varepsilon}{3K}$

By using lema (1.1), we obtain that $I_1<\varepsilon$.

For I_2 we have:

$$I_2\le \int_t^\infty |X(t)P_2X^{-1}(s)||f(s,y(g(s)))-f(s,0)|ds+\int_t^\infty |X(t)P_2X^{-1}(s)||f(s,0)|ds\le$$

$$\le K\varphi(\|y\|)\Big(\int_{t_1}^\infty \lambda(s)^p\Big)^{\frac{1}{p}}+K\Big(\int_{t_1}^\infty |f(s,0)|^p\Big)^{\frac{1}{p}}.$$

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