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NEW SOLUTIONS FOR YANG-BAXTER SYSTEMS

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Abstract.We present the concepts of Yang-Baxter equation and its generalisation, the Yang-Baxter system. We construct new Yang-Baxter systems from algebra and bialgebra structures.

1.Introduction

In a previuos talk at ICTAMI-2002 conference, we introduced the concept of Yang-Baxter system. In this paper, which follows a talk at ICTAMI-2005 conference, we first review the concepts of Yang-Baxter equation and its generalisation, the Yang-Baxter system. We present briefly the concepts of algebras, coalgebras and bialgebras. [12] and [4]constructed Yang-Baxter operators from algebras and coalgebras. The following question arises: What is the relation between those operators if we start with a bialgebra ? One answer is that they are connected via a Yang-Baxter system (see theorem 6.2). Another Yang-Baxter system is constructed directly from an algebra structure.

2.The Yang-Baxter equation

The Yang-Baxter equation first appeared in theoretical physics and statistical mechanics. Afterwards, it has proved to be important in knot theory, quantum groups, the quantization of integrable non-linear evolution systems, etc.

Throughout this paper k is a field. All tensor products appearing in this paper are defined over k.

Let V be a k-space. We denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$.

We use the following terminology concerning the Yang-Baxter equation.

Some references on this topic are: [8], [9],[10],[11] etc.

Let $R : V \otimes V \to V \otimes V$ be a k-linear map. We use the following notations: $R^{12} = R \otimes I, R^{23} = I \otimes R, R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$, where I_V or simply I is the identity map of the space V.

Definition 2.1 *An invertible k-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator (or simply a YB operator) if it satisfies the equation*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

Remark 2.2. *The equation (1) is usually called the braid equation. It is a well-known fact that the operator R satisfies (1) if and only if $R \circ \tau$ satisfies the quantum Yang-Baxter equation (if and only if $\tau \circ R$ satisfies the quantum Yang-Baxter equation):*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2}$$

Remark 2.3. *i) $\tau : V \otimes V \to V \otimes V$ is an example of a YB operator.*

ii) An exhaustive list of invertible solutions for (2) in dimension 2 is given in [5].

iii) Finding all Yang-Baxter operators in dimension greater then 2 is an unsolved problem.

3. Yang-Baxter systems

It is convenient to introduce the following notation from [7]: *the Yang-Baxter commutator [R,S,T]* of the maps $R : V \otimes V' \to V \otimes V'$, $S : V \otimes V'' \to V \otimes V''$ and $T : V' \otimes V'' \to V' \otimes V''$ is a map $[R, S, T] : V \otimes V' \otimes V'' \to V \otimes V' \otimes V''$, such that

$$[R, S, T] = R^{12} \circ S^{13} \circ T^{23} - T^{23} \circ S^{13} \circ R^{12} \ . \tag{3}$$

In this notation the quantum Yang-Baxter equation is written as:

$$[R, R, R] \ = \ 0 \ . \tag{4}$$

Definition 3.1. *The following system of equations is called a* WXZ system *(or a Yang-Baxter system):*

$$[W, W, W] = 0, \tag{5}$$

$$[Z, Z, Z] = 0, \tag{6}$$

$$[W, X, X] = 0, \tag{7}$$

$$[X, X, Z] = 0. \tag{8}$$

where $W : V \otimes V \to V \otimes V$, $Z : V' \otimes V' \to V' \otimes V'$ *and* $X : V \otimes V' \to V \otimes V'$.

Remark 3.2. A *WXZ system* is a constant version of the spectral dependent Yang-Baxter systems for nonultralocal models presented in [6].

Remark 3.3. A *WXZ system* is also related to the method of obtaining the quantum doubles for pairs of FRT quantum groups (see [15]).

Remark 3.4. *From a* WXZ system *with X invertible, one can construct a Yang-Baxter operator (see theorem 2.7 of [11]).*

Remark 3.5. *For examples and the classification of* WXZ systems *in dimension two (*$dim_k V = dim_k V' = 2$*), see [7].*

4.Algebras, coalgebras and bialgebras

In this section we present briefly the concepts of algebras, coalgebras and bialgebras. For more details we refer to [1], [3] or [14].

Definition 4.1.*A k*-algebra *is a k-space A with k-linear maps $M : A \otimes A \to A$ and $u : k \to A$ called* (associative) product *and* unit*, respectively, with properties* $M \circ (M \otimes I_A) = M \circ (I_A \otimes M)$*, and* $M \circ (I_A \otimes u) = I_A = M \circ (u \otimes I_A)$.

Definition 4.2.*A k*-coalgebra *is a k-space C with k-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ called* (coassociative) coproduct *and* counit*, respectively, with properties* $(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta$*, and* $(I_C \otimes \epsilon) \circ \Delta = I_C = (\epsilon \otimes I_C) \circ \Delta$.

Example. Let S be a set. Let kS be a k-space with S as a basis. Define $\Delta : kS \to kS \otimes kS,\ \Delta(s) = s \otimes s\ \forall s \in S,\ \epsilon : kS \to k,\ \epsilon(s) = 1\ \forall s \in S$. Then kS is a coalgebra.

Notation. For C a coalgebra and $c \in C$, we use Sweedler's notation: $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$.

Definition 4.3.*A k-space B that is an algebra $(B,\ M,\ u)$ and a coalgebra $(B,\ \Delta,\ \epsilon)$ is called a bialgebra if Δ and ϵ are algebra morphisms or, equivalently, M and u are coalgebra morphisms.*

5.Yang-Baxter operators from (co)algebra structures

Let A be a k-algebra, and $\alpha, \beta, \gamma \in k$. We define the k-linear map:

$R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A,\ \ R^A_{\alpha,\beta,\gamma}(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b.$

Theorem 5.1. (S. Dăscălescu and F. F. Nichita, [4]) *Let A be a k-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^A_{\alpha,\beta,\gamma}$ is a YB operator if and only if one of the following holds:*
(i) $\alpha = \gamma \neq 0,\ \ \beta \neq 0$;
(ii) $\beta = \gamma \neq 0,\ \ \alpha \neq 0$;
(iii) $\alpha = \beta = 0,\ \ \gamma \neq 0$.
If so, we have $(R^A_{\alpha,\beta,\gamma})^{-1} = R^A_{\frac{1}{\beta},\frac{1}{\alpha},\frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^A_{0,0,\gamma})^{-1} = R^A_{0,0,\frac{1}{\gamma}}$ in case (iii).

Remark 5.2.*The previous theorem can be transfered to coalgebras (see [4]).*

Let C be a k-coalgebra with $\dim C \geq 2$, and $\alpha, \beta, \gamma \in k$. We define the k-linear map $R^C_{\alpha,\beta,\gamma} : C \otimes C \to C \otimes C,\ \ R^C_{\alpha,\beta,\gamma}(c \otimes d) = \alpha\epsilon(d)\Delta(c) + \beta\epsilon(c)\Delta(d) - \gamma c \otimes d$.

Then $R^C_{\alpha,\beta,\gamma}$ is a YB operator if and only if one of the following holds:
(i) $\alpha = \gamma \neq 0,\ \ \beta \neq 0$;
(ii) $\beta = \gamma \neq 0,\ \ \alpha \neq 0$;

(iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.
If so, we have $(R^C_{\alpha,\beta,\gamma})^{-1} = R^C_{\frac{1}{\beta},\frac{1}{\alpha},\frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^C_{0,0,\gamma})^{-1} = R^C_{0,0,\frac{1}{\gamma}}$ in case (iii).

6. Yang-Baxter systems from algebra and bialgebra structures

Theorem 6.1. (F. F. Nichita and D. Parashar, [13]) *Let A be a k-algebra, and $\lambda, \mu \in k$. The following is a Yang-Baxter system:*
$W : A \otimes A \to A \otimes A, \quad W(a \otimes b) = ab \otimes 1 + \lambda 1 \otimes ab - b \otimes a,$
$Z : A \otimes A \to A \otimes A, \quad Z(a \otimes b) = \mu ab \otimes 1 + 1 \otimes ab - b \otimes a,$
$X : A \otimes A \to A \otimes A, \quad X(a \otimes b) = ab \otimes 1 + 1 \otimes ab - b \otimes a.$

Theorem 6.2. *Let B be a k-bialgebra, and $r, s, p, t \in k$. The following is a Yang-Baxter system:*
$W : B \otimes B \to B \otimes B, \quad W(a \otimes b) = sba \otimes 1 + r1 \otimes ba - sb \otimes a$
$X : B \otimes B \to B \otimes B, \quad X(a \otimes c) = \sum_a a_1 \otimes ca_2$
$Z : B \otimes B \to B \otimes B, \quad Z(b \otimes c) = t\epsilon(b) \sum_{(c)} c_1 \otimes c_2 + p\epsilon(c) \sum_{(b)} b_1 \otimes b_2 - pc \otimes b$

Proof. We present a direct proof. Another proof can be obtained as a consequence of the theory developed in [2].

$[W, W, W] = 0$ and $[Z, Z, Z] = 0$ follow from section 5.

$[W, X, X] = 0 \iff W^{12} \circ X^{13} \circ X^{23} = X^{23} \circ X^{13} \circ W^{12}$

$W_{12} \circ X_{13} \circ X_{23}(a \otimes b \otimes c) = W_{12} \circ X_{13}(\sum_{(b)} a \otimes b_1 \otimes cb_2) = W_{12}(\sum_{(a),(b)} a_1 \otimes b_1 \otimes (cb_2)a_2) = s \sum_{(a),(b)} b_1 a_1 \otimes 1 \otimes (cb_2)a_2 + r \sum_{(a),(b)} 1 \otimes b_1 a_1 \otimes (cb_2)a_2 - s \sum_{(a),(b)} b_1 \otimes a_1 \otimes (cb_2)a_2$

$X_{23} \circ X_{13} \circ W_{12}(a \otimes b \otimes c) = X_{23} \circ X_{13}(sba \otimes 1 \otimes c + r1 \otimes ba \otimes c - sb \otimes a \otimes c) = X_{23}(s \sum_{(ba)} (ba)_1 \otimes 1 \otimes c(ba)_2 + r1 \otimes ba \otimes c - s \sum_{(b)} b_1 \otimes a \otimes cb_2) = s \sum_{(ba)} (ba)_1 \otimes 1 \otimes c(ba)_2 + r \sum_{(ba)} 1 \otimes (ba)_1 \otimes c(ba)_2 - s \sum_{(a),(b)} b_1 \otimes a_1 \otimes (cb_2)a_2 = s \sum_{(a),(b)} b_1 a_1 \otimes 1 \otimes c(b_2 a_2) + r \sum_{(a),(b)} 1 \otimes b_1 a_1 \otimes c(b_2 a_2) - s \sum_{(a),(b)} b_1 \otimes a_1 \otimes (cb_2)a_2$
The last equality holds because we work with a bialgebra.

Thus, $W_{12} \circ X_{13} \circ X_{23}(a \otimes b \otimes c) = X_{23} \circ X_{13} \circ W_{12}(a \otimes b \otimes c)$

$[X,X,Z]=0 \iff X^{12}\circ X^{13}\circ Z^{23}=Z^{23}\circ X^{13}\circ X^{12}$

$X^{12}\circ X^{13}\circ Z^{23}(a\otimes b\otimes c)=X^{12}\circ X^{13}(t\epsilon(b)\sum_{(c)}a\otimes c_1\otimes c_2+p\epsilon(c)\sum_{(b)}a\otimes b_1\otimes b_2-pa\otimes c\otimes b)=X^{12}(t\epsilon(b)\sum_{(a),(c)}a_1\otimes c_1\otimes c_2a_2+p\epsilon(c)\sum_{(a),(b)}a_1\otimes b_1\otimes b_2a_2-p\sum_a a_1\otimes c\otimes ba_2)=t\epsilon(b)\sum_{(a),(c)}a_1\otimes c_1a_2\otimes c_2a_3+p\epsilon(c)\sum_{(a),(b)}a_1\otimes b_1a_2\otimes b_2a_3-p\sum_a a_1\otimes ca_2\otimes ba_3$

$Z^{23}\circ X^{13}\circ X^{12}(a\otimes b\otimes c)=Z^{23}\circ X^{13}(\sum_{(a)}a_1\otimes ba_2\otimes c)=Z^{23}(\sum_{(a)}a_1\otimes ba_3\otimes ca_2)=t\epsilon(ba_4)\sum_{(a),(c)}a_1\otimes c_1a_2\otimes c_2a_3+p\epsilon(ca_2)\sum_{(a),(b)}a_1\otimes b_1a_3\otimes b_2a_4-p\sum_{(a)}a_1\otimes ca_2\otimes ba_3=t\epsilon(b)\sum_{(a),(c)}a_1\otimes c_1a_2\otimes c_2a_3+p\epsilon(c)\sum_{(a),(b)}a_1\otimes b_1a_2\otimes b_2a_3-p\sum_a a_1\otimes ca_2\otimes ba_3$

The last equality holds because we work with a bialgebra.

Thus, $X^{12}\circ X^{13}\circ Z^{23}(a\otimes b\otimes c)=Z^{23}\circ X^{13}\circ X^{12}(a\otimes b\otimes c)$.

Remark 6.3. In theorem 6.2, if B is a Hopf algebra then X is invertible. A large class of Yang-Baxter operators can be obtained in this case using remark 3.4.

Remark 6.4. Theorem 6.2 was generalised in [2]. Thus, one can construct Yang-Baxter systems from entwining structures. A reciprocal of this theorem also works.

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