ACTA UNIVERSITATIS APULENSIS No 11/2006

Proceedings of the *International Conference on Theory and Application of Mathematics and Informatics ICTAMI 2005* - Alba Iulia, Romania

CONSTRUCTION OF STANCU-HURWITZ OPERATOR FOR TWO VARIABLES

Ioana Taşcu and Andrei Horvat-Marc

2000 Mathematics Subject Classification: 41A10, 41A36

Keywords and phrases:Hurwitz's identity, linear positive operator of Stancu-Hurwitz type.

Abstract.In this paper we construct and study a linear bivariate operator of Stancu-Hurwitz type. The approximation formula has the degree of exactness (1) and for the remainder we use a method of T. Popoviciu.

1.Introduction

In a paper published in 1902 by Hurwitz A. [3] there has been given an identity, generalizing the classical identity of Abel - Jensen, namely

$$\begin{gathered}(u+v)(u+v+\beta_1+\ldots+\beta_m)^{m-1}= \\ =\sum u(u+\beta_{i_1}+\ldots+\beta_{i_k})^{k-1}v(v+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1},\end{gathered} \tag{1}$$

where $\beta_1,\ldots,\beta_m$ are m nonnegative parameters.

In the special case $\beta_1=\beta_2=\ldots=\beta_m=\beta$, this identity reduces to the case of Abel-Jensen:

$$(u+v)(u+v+m\beta)^{m-1}=\sum_{k=0}^{m}\binom{m}{k}u(u+k\beta)^{k-1}v(v+(m-k)\beta)^{m-k-1}.$$

Replacing in (1) $u=x$ and $v=1-x$, the identity becomes

$$\begin{gathered}(1+\beta_1+\ldots+\beta_m)^{m-1}= \\ =\sum x(x+\beta i_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.\end{gathered} \tag{2}$$

Using this relation in a paper published in 2002 [11], D. D. Stancu construct an approximation linear positive operator, depending on m nonnegative parameters $\beta_1,\ldots,\beta_m$, defined for any function $f\in C[0,1]$, denoted by $S_m^{(\beta_1,\ldots,\beta_m)}$, having the following formula

$$\left(S_m^{(\beta_1,\ldots,\beta_m)}f\right)(x)=\sum_{k=0}^{m} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)f\left(\frac{k}{m}\right), \tag{3}$$

where

$$\begin{gathered}(1+\beta_1+\ldots+\beta_m)^{m-1}w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)=\\ =\sum x(x+\beta_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.\end{gathered} \tag{4}$$

This operator is interpolatory at both sides of the interval $[0,1]$ and reproduces the linear functions.

2.Construction of Stancu-Hurwitz operator for two variables

We consider the space of real-valued bivariate functions $C(D)$, where D is the unit square $[0,1]\times[0,1]$ and we construct by a tensor product a bivariate polynomials of Stancu-Hurwitz type given by the formula

$$\left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y)=\sum_{k=0}^{m}\sum_{\gamma=0}^{n} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y)f\left(\frac{k}{m},\frac{\nu}{n}\right), \tag{5}$$

where

$$\begin{gathered}(1+\beta_1+\ldots+\beta_m)^{m-1}w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)\\ =\sum x(x+\beta i_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}\end{gathered} \tag{6}$$

$$\begin{gathered}(1+\gamma_1+\ldots+\gamma_n)v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y)\\ =\sum y(y+\gamma_{s_1}+\ldots+\gamma_{s_\nu})^{\nu-1}(1-y)(1-y+\gamma_{t_1}+\ldots+\gamma_{t_{n-\nu}})^{n-\nu-1}\end{gathered} \tag{7}$$

and $\beta_1,\ldots,\beta_m,\gamma_1,\ldots,\gamma_n$ are nonnegative parameters.

This operators represents an extension to two variables of the second operators of Cheney-Sharma [1] because for the special cases $\beta_1=\ldots=\beta_m=\beta$ and $\gamma_1=\ldots=\gamma_n=\gamma$ we obtain the Cheney-Sharma-Stancu operator

$$\left(S_{m,n}^{(\beta,\gamma)}f\right)(x,y)=\sum_{k=0}^{m}\sum_{\nu=0}^{n} w_{m,k}^{(\beta)}(x)v_{n,\nu}^{(\gamma)}(y)f\left(\frac{k}{m},\frac{\nu}{n}\right)$$

where

$$(1+m\beta)^{m-1}w_{m,k}^{(\beta)}(x)=\binom{m}{k}x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-k-1}$$

$$(1+n\gamma)^{n-1}v_{n,\nu}^{(\gamma)}(y)=\binom{n}{\gamma}y(y+\nu\gamma)^{\nu-1}(1-y)(1-y+(n-\nu)\gamma)^{n-\nu-1}$$

THEOREM 2.1. *The polynomial defined at (5) is interpolatory in the corners of the square D and the approximation formula*

$$f(x,y)=\left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y)+\left(R_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y) \tag{8}$$

has the degree of exactness $(1,1)$.

Proof. We can write

$$\begin{gathered}(1+\beta_1+\ldots+\beta_m)^{m-1}(1+\gamma_1+\ldots+\gamma_n)^{n-1}\left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y)\\ =(1-x)(1-x+\beta_1+\ldots+\beta_m)^{m-1}(1-y)(1-y+\gamma_1+\ldots+\gamma_n)^{n-1}f(0,0)\\ +(1-x)(1-x+\beta_1+\ldots+\beta_m)^{m-1}y(y+\gamma_1+\ldots+\gamma_n)^{n-1}f(0,1)\\ +x(x+\beta_1+\ldots+\beta_m)^{m-1}(1-y)(1-y+\gamma_1+\ldots+\gamma_n)^{n-1}f(1,0)\\ +x(1-x)y(1-y)\sum_{k=1}^{m-1}\sum_{\nu=1}^{n-1}w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)v_{n,\nu}^{(\gamma_1,\ldots,\gamma_n)}(y)f\left(\frac{k}{m},\frac{\nu}{n}\right)\\ +x(x+\beta_1+\ldots+\beta_m)^{m-1}y(y+\gamma_1+\ldots+\gamma_n)^{n-1}f(1,1)\end{gathered} \tag{9}$$

and we can see that the polynomial reproduces the values of the function $f\in C(D)$ in the corners of the square $D:(0,0),(1,0),(0,1),(1,1)$. Hence

$$\begin{gathered}\left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(0,0)=f(0,0)\\ \left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(1,0)=f(1,0)\\ \left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(0,1)=f(0,1)\\ \left(S_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(1,1)=f(1,1)\end{gathered} \tag{10}$$

it follows that the operator reproduces the linear functions and the approximation formula (8) has the degree of exactness $(1,1)$.

If we apply a criterion of T. Popoviciu [7], we find that the remainder of the approximation formula (5) is of simple form and we can state

THEOREM 2.2. *If the function f has continuous second-order partial derivatives on the square D, then the remainder of the approximation formula (5) can be represented under the form:*

$$\begin{aligned}&\left(R_{m,n}^{(\beta_1,\ldots,\beta_m;\gamma_1,\ldots,\gamma_n)}f\right)(x,y)\\ =&\ \tfrac{1}{2}\left(R_m^{(\beta_1,\ldots,\beta_m)}e_{2,0}\right)(x)f^{(2,0)}(\xi,y)+\tfrac{1}{2}\left(R_n^{(\gamma_1,\ldots,\gamma_n)}e_{0,2}\right)(y)f^{(0,2)}(x,\eta)\\ &-\tfrac{1}{4}\left(R_m^{(\beta_1,\ldots,\beta_m)}e_{2,0}\right)(x)\left(R_n^{(\gamma_1,\ldots,\gamma_n)}e_{0,2}\right)(y)f^{(2,2)}(\xi,\eta),\end{aligned} \tag{11}$$

where $e_{2,0}(x,y)=x^2$, $e_{0,2}(x,y)=y^2$.

Using a theorem of Peano-Milne-Stancu type the remainder can have an integral representation.

REFERENCES

[1] Cheney, E. W., Sharma, A., *On a generalization of Bernstein polynomials*, Rev. Mat. Univ. Parma, **5** (1964), 77-84.

[2] Horova, I., Budikova, M., *A note on D. D. Stancu operators*, Ricerche di Matem., **44** (1995), 397-407.

[3] Hurwitz, A., *Über Abels Verallgemeinerung der binomischen Formel*, Acta Math., **26** (1902), 199-203.

[4] Jensen, L. W., *Sur une identité d'Abel et sur d'autres formules analogues*, Acta Math., **26** (1902), 307-318.

[5] Milne, W. E., *The remainder in linear methods of approximation*, J. Res. Mat. Bur. Standards, **43** (1949), 501-511.

[6] Peano, G., *Resto nelle formule di quadratura expresso con un integrale definito*, Atti Acad. Naz. Lincei Rend., **22** (1913), 562-569.

[7] Popoviciu, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica, Cluj, **1**(**24**) (1959), 95-142.

[8] Stancu, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17** (1963), 270-278.

[9] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. B, **1** (1964), 137-163.

[10] Stancu, D. D., *Methods for construction of linear positive operators of approximation*, Numerical Analysis and Approximation Theory, Proc. of the International Symposium, edit. R. T. Trîmbiţaş, Cluj University Press, 2002, 23-45.

[11] Stancu, D. D., *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Analyse Numér. Théorie de l'approximation, Cluj, **31** (2002), 115-118.

[12] Stancu, D. D., Cişmaşiu, C., *On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numer. Cluj, **26** (1997), 221-227.

Tascu Ioana and Andrei Horvat-Marc
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 43012 Baia Mare, Romania
email:itascu@yahoo.com
email:hmandrei@yahoo.com