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ON GENERALIZED PROBABILISTIC 2-NORMED SPACES

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Abstract. In [1] C. Alsina, B. Schweizer and A. Sklar gave a new definition of a probabilistic normed space. This definition, is based on a characterization of normed spaces by a betweenness relation and put the theory of probabilistic normed spaces on a new general basis. Starting from this idea we study from a new and more general point of view the probabilistic 2-normed spaces. Topological properties and examples for these generalized probabilistic 2-normed spaces are given. A new point of view for practical applications is considered.

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1.Introduction

The notion of probabilistic metric space, was introduced in [18] by replacing the real values of a metric by probability distribution functions. This new point of view led to a large development of probabilistic analysis. Applications to systems having hysteresis, mixture processes, the measuring errors theory were also given [3], [13], [20].

In [21] A. N. Šerstnev used the K. Menger's idea and endowed a set having an algebraic structure of linear space with a probabilistic norm and set the bases of the probabilistic normed space theory.

The notions of a 2-metric space and of a linear 2-normed space were first introduced by S. Gähler in [6]and [7], respectively. Since then, the theory of 2-metric spaces and of 2-normed spaces were enhanced and deep studies were made, we refer [4],[5],[15].

In some papers the probabilistic 2-metric spaces and probabilistic 2-normed spaces were also considered and some results were obtained [8-9], [12].

As usual $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and $I = [0, 1]$ is the closed unit interval. Let Δ^+ denote the set of all distance distribution functions, i.e., the set of non-decreasing, left-continuous functions $F : \mathbb{R} \to I$ with $F(0) = 0$.

The set Δ^+ will be endowed with the topology given by the modified Levy metric d_L [20]. Let $\mathbb{D}^+$ denote the set of those functions $F \in \Delta^+$ for which $\lim\limits_{t\to\infty} F(t) = 1$. If $F, G \in \Delta^+$, then we write $F \leq G$ if $F(t) \leq G(t)$, for all $t \in \mathbb{R}$.

If $a \in \mathbb{R}^+$, then H_a will be the special function of Δ^+, defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$. By the injective map $a \to H_a$ $\mathbb{R}^+$ can be viewed as a subspace of $\mathbb{D}^+$. It is obvious that $H_0 \geq F$, for all $F \in \Delta^+$.

Recall that a t-norm is a two place map $T : I \times I \to I$ that is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$. A mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ is a triangle function if it is commutative, associative and it has the H_0 as the identity, i.e. $\tau(F, H_{0)}) = F$, for every $F \in \Delta^+$. Note that, if T is a left continuous t-norm and τ_T is defined by :

$(T)\quad \tau_T(F, G)(t) = \sup\limits_{t_1+t_2<t} T(F(t_1), G(t_2)),$

then τ_T is a triangle function [20].

Let T_1 be a t-norm, then the function $T : I \times I \times I \longrightarrow I$ defined by $T(a, b, c) = T_1(a, T_1(b, c))$ is called a th-norm. If τ_1 is a triangle function, then the function $\tau : \Delta^+ \times \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ defined by $\tau(F, G, H) = \tau_1(F, \tau_1(G, H))$, is called a tetrahedral function (in brief th-function). These functions possess appropriate properties to define tetrahedral inequality in probabilistic 2-metric spaces.

Definition 1. *A probabilistic normed space is a ordered triple $(L, \mathcal{F}, \tau)$, where L is a real linear space, τ is a continuous triangle functions, and $\mathcal{F}$ is a mapping from L into D^+ such that, for all $x, y \in L$ and $\alpha \in \mathbb{R}$ the following conditions hold :*
(S_1) $F_x = H_0$ if and only if $x = \theta$.
(S_2) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$
(S_3) $F_{x+y} \geq \tau(F_x, F_y)$.

We adopt the convention that $F(\frac{t}{0}) = H_0(t)$, for all $t > 0$. The theory of probabilistic normed spaces has been developed in concordance with that of ordinary normed spaces and with that of probabilistic metric spaces. It is important in their own right and it also give new tools in the study of random operator equations. For important results of probabilistic functional analysis we refer [2], [3], [11], [20].

A new definition of a probabilistic normed space has been given in [1] and it incudes the above Šerstnev's definition as a special case.

Definition 2. *A probabilistic normed space is a quadruple* $(L, \mathcal{F}, \tau, \tau^*)$*, where* L *is a real linear space,* τ *and* τ^* *are continuous triangle functions with* $\tau \leq \tau^*$*, and* $\mathcal{F}$ *is a mapping from* L *into* D^+ *such that, for all* $x, y \in L$ *the conditions* (S_1) *and* (S_3) *hold and the condition* (S_2) *is replaced by :*
(S_4) $F_{-x} = F_x$;
(S_5) $F_x \leqslant \tau^*(F_{\alpha x}, F_{(1-\alpha)x})$, *for* $\alpha \in [0,1]$.
This definition is regarded as both natural and fruitful. One seems that it has a great potential for future applications to various field of mathematics and other areas.

2.Generalized probabilistic 2-normed spaces

In the this section we give an enlargement by two ways of probabilistic 2-normed spaces.

Definition 3. *A probabilistic 2-metric space (briefly, a P-2M space) is a triple* $(S, \mathcal{F}, \tau)$*, where* S *is a nonempty set whose elements are the points of the space,* $\mathcal{F}$ *is a mapping from* $S \times S \times S$ *into* $D^+, \mathcal{F}(x, y, z)$ *will be denote by* $F_{x,y,z}$*,* τ *is a tetrahedral function and the following conditions are satisfied, for all* $x, y, z, u \in S$.
(P_1) *To each pair of distinct points* x, y *in* S *there exists a point* z *in* S *such that* $F_{x,y,z} \neq H_0$.
(P_2) $F_{x,y,z} = H_0$ *if at least two of* x, y, z *are equal.*
(P_3) $F_{x,y,z} = F_{x,z,y} = F_{y,z,x}$.
(P_4) $F_{x,y,z} \geq \tau(F_{x,y,u}, F_{x,u,z}, F_{u,y,z})$.

Let T be a th-norm and let us consider the following inequality :
(P_5) $F_{x,y,z}(t) \geq T(F_{x,y,u}(t_1), F_{x,u,z}(t_2), F_{u,y,z}(t_3)), \quad (\forall) \quad t_1, t_2, t_3 \in \mathbb{R}^+$
$t_1 + t_2 + t_3 = t$.
If (P_4) is replaces by (P_5) then the triple $(S, \mathcal{F}, T)$ is called a P-2M space of Menger's type or simply a 2-Menger space.

Definition 4. *Let* L *be a linear space of a dimension greater than one,* τ *a triangle function , and let* $\mathcal{F}$ *be a mapping from* $L \times L$ *into* D^+*. If the*

following conditions are satisfied :
$(N1)$ $F_{x,y} = H_0$ *if* x *and* y *are linearly dependent,*
$(N2)$ $F_{x,y} \neq H_0$ *if* x *and* y *are linearly independent,*
$(N3)$ $F_{x,y} = F_{y,x}$, *for every* x, y *in* L,
$(N4)$ $F_{\alpha x,y}(t) = F_{x,y}(\frac{t}{|\alpha|})$, *for every* $t > 0, \alpha \neq 0$ *and* $x, y \in L$,
$(N5)$ $F_{x+y,z} \geq \tau(F_{xz}, F_{yz})$, *whenever* $x, y, z \in L$, ,
then $\mathcal{F}$ *is called a probabilistic 2-norm on* L *and the triple* $(L, \mathcal{F}, \tau)$ *is called a probabilistic 2-normed space (briefly P-2N space)* [9].

If the triangle inequality (N5) is formulated under a t-norm T :

$(N5')$ $F_{x+y,z}(t_1 + t_2) \geq T(F_{xz}(t_1), F_{yz}(t_2))$, for all $x, y, z, \in L, t_1, t_2 \in \mathbb{R}_+$,
then the triple $(L, \mathcal{F}, T)$ is called a Menger 2-normed space.
If T is a left continuous t-norm and τ_T is the associated triangle function, then the inequalities $(N5)$ and $(N5')$ are equivalent.

Now, we will give an enlargement of the notion of probabilistic 2-normed space by generalizing the axiom which give a connection between the distribution functions of a vector and its product by a real number.

Let φ be a function defined on the real field $\mathbb{R}$ into itself, with the following properties : (a) $\varphi(-t) = \varphi(t)$, for every $t \in \mathbb{R}$; (b) $\varphi(1) = 1$; (c) φ is strict increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim\limits_{\alpha \to \infty} \varphi(\alpha) = \infty$. Examples of such functions are : $\varphi(\alpha) = |\alpha|$; $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$; $\varphi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}^+$.

Definition 5. *Let* L *be a linear space of a dimension greater than one,* τ *a triangle function , and let* $\mathcal{F}$ *be a mapping from* $L \times L$ *into* D^+. *If the conditions* $(N1)$, $(N2)$, $(N3)$, *and* $(N5)$ *are satisfied and the condition* $(N4)$ *is replaced by :*
$(N4')$ $F_{\alpha x,y}(t) = F_{x,y}(\frac{t}{\varphi(\alpha)})$, *for every* $t > 0, \alpha \neq 0$ *and* $x, y \in L$,
then the triple $(L, \mathcal{F}, \tau)$ *is called a probabilistic* φ*-2-normed space.* *We recall that a 2-normed space is a pair* $(L, ||\cdot, \cdot||)$ ([7]), *where* L *is a linear space of a dimension greater than one and* $||\cdot, \cdot||$ *is a real valued mapping on* $L \times L$, *such that the following conditions be satisfied :*
(G₁) $||x, y|| = 0$ *if, and only if,* x *and* y *are linearly dependent,*
(G2) $||x, y|| = ||y, x||$, *for all* $x, y \in L$,
(G₃) $||\alpha \cdot x, y|| = |\alpha| ||x, y||$, *whenever* $x, y \in L$ *and* $\alpha \in \mathbb{R}$,
(G₄) $||x + y, z|| \leq ||x, z|| + ||y, z||$, *for all* $x, y, z \in L$.

If the conditions $(G1)$, $(G2)$, and $(G4)$ are satisfied and the condition $(G3)$ is replaced by :
$(G3')$ $||\alpha x, y|| = \varphi(\alpha)||x, y||$, whenever $x, y \in L$ and $\alpha \in \mathbb{R}$
then, the pair $(L, ||\cdot, \cdot||)$ is called a a φ-2-normed space.

Remark 1. *It is easy to check that every φ-2-normed space $(L, ||\cdot, \cdot||)$ can be made a probabilistic φ-2-normed space, in a natural way, by setting $F_{x,y}(t) = H_0(t - ||x, y||)$, for every $x, y \in L$, $t \in \mathbb{R}_+$ and $T = Min$.*

Proposition 1. *Let $G \in D^+$ be different from H_0, let $(L, ||\cdot, \cdot||)$ be a φ-2-normed space with $\varphi(\alpha) = |\alpha|^p$, $p \in (0, 1]$, and $\mathcal{F} : L \times L \to D^+$ defined by $F_{\theta,x} = H_0$ and, if $x, y \neq \theta$ by*

$$F_{x,y}(t) = G(\frac{t}{||x, y||}), \quad t \in \mathbb{R}_+.$$

Then, the triple $(L, \mathcal{F}, T)$ becomes a probabilistic φ-2-normed space under the t-norm $T = Min$ and the $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$. This is called a simple probabilistic φ-2-normed space generated by the distribution function G and the φ-2-normed space $(L, ||\cdot, \cdot||)$.

Proof. Let us verify the axioms $(N4')$ and $(N5')$. Indeed

$$F_{\alpha x,y}(t) = G(\frac{t}{||\alpha x, y||^p}) = G(\frac{t}{|\alpha|^p||x, y||}) = F_{x,y}(\frac{t}{|\alpha|^p}).$$

In order to prove the inequality $(N5)$ we shall use the properties of a quasi-inverse $F^\wedge$ of a distribution function F defined by : $F^\wedge(u) = \sup\ \{t : F(t) < u\}$. Since $F^\wedge_{x,y} = ||x, y||G^\wedge$, we have, for every $x, y, z \in L$,

$$[\tau_M(F_{x,z}, F_{y,z}]^\wedge = F^\wedge_{x,z} + F^\wedge_{y,z} = ||x, z||G^\wedge + ||y, z||G^\wedge \geq ||x + y, z||G^\wedge = F^\wedge_{x+y,z}.$$

But, the above inequality is equivalent with $F_{x+y,z} \geqslant \tau_M(F_{x,z}, F_{y,z})$. So, the inequality $(N5)$ is satisfied.

The above proposition shows that, by starting from a 2-normed space or from a φ-2-normed space, for particular functions G, different probabilistic φ-2-normed spaces can be obtained. So, probabilistic φ-2-normed spaces have a more large statistical disposal. So, every process of measurement of a pair of vectors can be statistical interpreted by using an appropriate statistical distribution function G.

The following theorem give a topological structure of probabilistic φ-2-normed spaces. Let $(L, \mathcal{F}, \tau)$ be a probabilistic φ-2-normed space and $\mathcal{A}$ be the family of all finite and non-empty subsets of the linear space L. For every $A \in \mathcal{A}$, $\varepsilon > 0$ and $\lambda \in (0,1)$ we define a neighborhood of the origin as being the subset of L given by

$$(V) \qquad V(\varepsilon, \lambda, A) = \{x \in L : F_{x,a}(\varepsilon) > 1-\lambda, a \in A\}$$

Theorem 1. *Let $(L, \mathcal{F}, \tau)$ be a probabilistic φ-2-normed space under a continuous triangle function τ such that $\tau \geq \tau_{T_m}$, where $T_m(a,b) = \max\{a+b-1, 0\}$, then $(L, \mathcal{F}, \tau)$ becomes a Hausdorff linear topological space having as a fundamental system of neighborhoods of the null vector θ the family :*

$$(F) \qquad \mathcal{V}_\theta = \{V(\epsilon, \lambda, A) : \varepsilon > 0, \lambda \in (0,1) A \in \mathcal{A}\}.$$

Proof. Let $V(\varepsilon_k, \lambda_k, A_k), k = 1,2$ be in $\mathcal{V}$. We consider $A = A_1 \cup A_2$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $\lambda = \min\{\lambda_1, \lambda_2\}$, then $V(\varepsilon, \lambda, A) \subset V(\varepsilon_1, \lambda_1, A_1) \cap V(\varepsilon_2, \lambda_2, A_2)$.

Let $\alpha \in \mathbb{R}$ such that $0 \leq |\alpha| \leq 1$ and $x \in \alpha V(\varepsilon, \lambda, A)$, then $x = \alpha y$, where $y \in V(\varepsilon, \lambda, A)$. For every $a \in A$ we have

$$F_{x,a}(\varepsilon) = F_{\alpha y,a}(\varepsilon) = F_{y,a}(\frac{\varepsilon}{\varphi(\alpha)}) \geq F_{y,a}(\varepsilon) > 1 - \lambda.$$

This shows us that $x \in V(\varepsilon, \lambda, A)$, hence $\alpha V(\varepsilon, \lambda, A) \subset V(\varepsilon, \lambda, A)$.

Now, let's show that, for every $V \subset \mathcal{V}$ and $x \in L$ there exists $\beta \in \mathbb{R}$, $\beta \neq 0$ such that $\beta x \in V$. If $V \in \mathcal{V}_M$ then there exists $\varepsilon > 0$, $\lambda \in (0,1)$ and $A \in \mathcal{A}$ such that $V = V(\varepsilon, \lambda, A)$. Let x be arbitrarily fixed in L and $\alpha \in \mathbb{R}, \alpha \neq 0$, then $F_{\alpha x,a}(\varepsilon) = F_{x,a}(\frac{\varepsilon}{\varphi(\alpha)})$. Since $\lim\limits_{|\alpha| \to 0} F_{x,a}(\frac{\varepsilon}{\varphi(\alpha)}) = 1$ it follows that, for every $a \in A$ there exists $\alpha(a) \in \mathbb{R}$ such that $F_{x,a}(\frac{\varepsilon}{\varphi(\alpha(a))}) > 1 - \lambda$. If we choose $\beta = \min\{|\alpha(a)| : a \in A\}$, then we have

$$F_{\beta x,a}(\varepsilon) = F_{x,a}(\frac{\varepsilon}{\varphi(\beta)}) \geq F_{x,a}(\frac{\varepsilon}{\varphi(\alpha(a))}) > 1 - \lambda,$$

for all $a \in A$, hence $\beta x \in V$.

Let us prove that, for any $V \in \mathcal{V}_M$ there exists $V_0 \in \mathcal{V}$ such that $V_0 + V_0 \subset V$.

If $V = V(\varepsilon, \lambda, A)$ and $x \in V(\varepsilon, \lambda, A)$, then there exists $\eta > 0$ such that

$F_{x,a}(\varepsilon) > 1-\eta > 1-\lambda$. If $V_0 = V(\frac{\varepsilon}{2}, \frac{\eta}{2}, A)$ and $x, y \in V_0$, $a \in A$ by triangle inequality we have

$$F_{x+y,a}(\varepsilon) \geq T(F_{x,a}(\frac{\varepsilon}{2}), F_{y,a}(\frac{\varepsilon}{2}))$$

$$\geq T(1-\frac{\eta}{2}, 1-\frac{\eta}{2}) \geq T_m(1-\frac{\eta}{2}, 1-\frac{\eta}{2}) > 1-\eta > 1-\lambda.$$

The above inequalities show us that $V_0 + V_0 \subset V$.

In that follows we show that $V \in \mathcal{V}$ and $\alpha \in \mathbb{R},\ \alpha \neq 0$ implies $\alpha V \in \mathcal{V}_M$.

Let us remark that $\alpha V = \alpha V(\varepsilon, \lambda, A) = \{\alpha x : F_{x,a}(\varepsilon) > 1-\lambda, a \in A)$ and $F_{x,a}(\varepsilon) > 1-\lambda \Leftrightarrow F_{x,a}(\frac{\varphi(\alpha)\varepsilon}{\varphi(\alpha)}) = F_{\alpha x,a}(\varphi(\alpha)\varepsilon) > 1-\lambda$. This shows that $\alpha V = V(\varphi(\alpha)\varepsilon, \lambda, A)$, hence $\alpha V \in \mathcal{V}$.

The above statements show us that $\mathcal{V}$ is a base of neighborhoods of the origin for a topology on the linear space L. This is generated by the probabilistic φ-2-norm $\mathcal{F}$ and is named $\mathcal{F}$-topology on L.

We now consider the following example of probabilistic φ-2-normed space having as base spaces sets of random variables with values in a Banach algebra.

The study of Banach algebra-valued random variables is of great importance in the theory of random equations since many of the Banach spaces encountered are also algebras.

Let $(X, \|.\|)$ be a separable Banach space which is also an algebra. Let $(\Omega, \mathcal{K}, P)$ be a complete probability measure space and let $(X, \mathcal{B})$ be the measurable space, where $\mathcal{B}$ is the σ -algebra of Borel subsets of the separable Banach algebra $(X, ||.||)$. We denote by L the linear space of all random variables defined on $(\Omega, \mathcal{K}, P)$ with values in $(X, \mathcal{B})$ equal with the probability one.

Since, in a Banach algebra, the operation of multiplication is continuous, the product of two X-valued random variables $x(\omega)y(\omega)$ is a well-defined X-valued random variable.

For all $x, y \in E$ and $t \in \mathbb{R}, t > 0$ we define

$$(B) \qquad \mathcal{F}_{x,y}(t) = F_{x,y}(t) = P(\{\omega \in \Omega : ||x(\omega)y(\omega)||^p < t\}),$$

where $p \in (0, 1]$.

Theorem 2. *The triple $(L, \mathcal{F}, T_m)$ is a probabilistic φ-2-normed space with $\varphi(\alpha) = |\alpha|^p$.*

Proof. We have to verify that conditions of Definition 5 are satisfied.
$F_{\alpha x,y}(t) = P(\{\omega \in \Omega : ||\alpha x(\omega)y(\omega)||^p < t\}) = P(\{\omega \in \Omega : \varphi(\alpha)||x(\omega)y(\omega)||^p <$

$t\}) = P(\{\omega \in \Omega : ||x(\omega)y(\omega)|| < \frac{t}{\varphi(\alpha}) = F_{x,y}(\frac{t}{\varphi(\alpha)})$. Similarly, one shows that $F_{x,\alpha y}(t) = F_{x,y}(\frac{t}{\varphi(\alpha)})$. So, the condition $(N4)$ is satisfied.

For each $x, y \in L, z \in M$, and $t_1, t_2 \in \mathbb{R}_+ - \{0\}$ we define the sets:

$$A = \{\omega \in \Omega : ||x(\omega)z(\omega)||^p < t_1\}, \quad B = \{\omega \in \Omega : ||y(\omega)z(\omega)||^p < t_2\},$$

$$C = \{\omega \in \Omega : ||[x(\omega) + y(\omega)]z(\omega)||^p < t_1 + t_2\}$$

From the triangle inequality of the norm $||.||$ it follows that $A \cap B \subset C$. By properties of the measure of probability P we have

$$P(C) \geq P(A \cap B) \geq P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1$$

Taking in account that $P(A) = F_{xz}(t_1) \quad P(B) = F_{y,z}(t_1)$ and $P(C) = F_{x+y,z}(t_1 + t_2)$, hence, the inequality $(N5)$ is satisfied.

Definition 6. *A generalized probabilistic 2-normed space is a quadruple $(L, \mathcal{F}, \tau, \tau^*)$, where L is a linear space of dimension greater than one, τ and τ^* are continuous triangle functions with $\tau \leqslant \tau^*$ and $\mathcal{F}$ is a mapping defined on $L \times L$ with values into D^+ such that for every x, y and z in L the following conditions hold :*

(A_1) $F_{x,y} = H_0$ *if, and only if, x and y are linearly dependent,*

(A_2) $F_{x,y} = F_{y,x}$,

(A_3) $F_{-x,y} = F_{x,y}$,

(A_4) $F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$,

(A_5) $F_{x,y} \leq \tau^*(F_{\alpha x,y}, F_{(1-\alpha)x,z})$, $\alpha \in [0, 1]$.

Proposition 2. *Every probabilistic 2-normed space is a generalized probabilistic 2-normed space, that is, if the condition (S_2) is satisfied then the conditions (A_3) and (A_5) are also satisfied.*

Theorem 3. *Let $(L, \mathcal{F}, \tau, \tau^*)$ be a generalized probabilistic 2-normed space and let*

$$\mathcal{M} : L \times L \times L \to D^+, \qquad M_{x,y,z} = F_{y-x,z-x},$$

then the triple $(L, \mathcal{M}, \tau)$ is a probabilistic 2-metric space.

Proof. Let's verify the axioms of a probabilistic 2-metric space.
From the assumption that L is of dimension greater than one, it follows that for every two distinct points $x, y \in L$ there exists $z \in L$ such that $z - x$ and

$y-x$ are linear independent, and therefore $M_{x,y,z} = F_{y-x,z-x} \neq H_0$. If at least two of the points x, y, z are equal then $y-x$ $z-x$ are linear dependent and we have $M_{x,y,z} = F_{y-x,z-x} = H_0$.
For every $x, y, z \in L$ $F_{y-x,z-x} = F_{z-x,y-x}$, therefore $M_{x,y,z} = M_{x,z,y}$. We also have $F_{z-x,y-x} = F_{[(z-y)-(x-y),-(x-y)]} \geqslant \tau(F_{z-y,x-y}, F_{x-y,x-y} \geqslant \tau(F_{z-y,x-y}, H_0) = F_{z-y,x-y}$. So, we have $F_{z-x,y-x} \geqslant F_{z-y,x-y}$. On the other hand $F_{z-y,x-y} = F_{[(z-x)-(y-x),-(x-y)]} \geqslant \tau(F_{z-x,y-x}, F_{x-y,x-y}) \geqslant \tau(F_{z-x,y-x}, H_0) = F_{z-x,y-x}$ and we also have $F_{z-y,x-y} \geqslant F_{z-x,y-x}$

By the above inequalities we have have $F_{z-y,x-y} = F_{z-x,y-x}$ which implies that $M_{x,y,z} = M_{y,z,x} = F_{x,z,y}$ for all $x, y, z \in L$.

Now, we prove the tetrahedral inequality in a probabilistic 2-metric space $(P4)$:

$$M_{x,y,z} = F_{y-x,z-x} = F_{[y-u-(x-u)],[z-u-(x-u)]} \geqslant$$
$$\tau(F_{y-u,z-u-(x-u)}, F_{-(x-u),z-u-(x-u)} \geqslant \tau(\tau(F_{y-u,z-u}, F_{y-u,x-u}), F_{x-u,z-u}) \geqslant$$
$$\tau(M_{x,y,z}, M_{x,u,z}, M_{x,y,u}).$$

So, the axioms of a probabilistic 2-metric space $(P1)-(P4)$ are satisfied.

THEOREM 4.*Let $(L, \mathcal{F}, \tau, \tau^*)$ be a generalized probabilistic 2-normed space. Then, the family (F) is a fundamental system of neighborhoods of the null vector in the linear space L.*
The proof of this theorem is similar to that of Theorem 1 and we have omitted it.

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