ACTA UNIVERSITATIS APULENSIS No 11/2006

Proceedings of the *International Conference on Theory and Application of Mathematics and Informatics ICTAMI 2005* - Alba Iulia, Romania

SOME RESULTS IN MULTIVARIATE INTERPOLATION

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Abstract. The aim of this paper is to introduce and analyze a multivariate interpolation scheme of Hermite-Birkhoff type. We found the interpolation space, the interpolation operator and its dual operator.

2000 *Mathematics Subject Classification*: 65D32

1. Introduction

The multivariate polynomial interpolation is very important for the practical applications.

Let $\mathcal{F}$ be a set of analytic functions which includes polynomials. In multivariate polynomial interpolation we must find a polynomial subspace, $\mathcal{P}$, such that, for a given set of functionals, Λ, there is an unique polynomial $p \in \mathcal{P}$ which matches an arbitrary function $f \in \mathcal{F}$, on every functional from Λ, that is

$$\lambda(f) = \lambda(p), \ \forall \lambda \in \Lambda \tag{1}$$

The pair $(\Lambda, \mathcal{P})$ represents an interpolation scheme. The space $\mathcal{P}$ is named an interpolation space for the set of conditions Λ. If there are not any other interpolation polynomial subspaces of degree less than the degree of the polynomials in $\mathcal{P}$, then the subspace $\mathcal{P}$ is a minimal interpolation space.

The multivariate interpolation problem is more difficult than the univariate interpolation problem. The first difficulty is given by the fact that the dimension of the subspace of polynomials of degree n does not cover the entire set of natural numbers. The dimension of the subspace of polynomials of degree n in d variables, Π_n^d, is

$$dim \ \Pi_n^d = \begin{pmatrix} n+d \\ d \end{pmatrix}, \tag{2}$$

so that, not for every set of functionals there is a natural number n such that

$$card(\Lambda) = dim \ \Pi_n^d \tag{3}$$

Even that the equality (3) is satisfied, Π_n^d is not an interpolation space for the conditions Λ if there is a polynomial $p \in \Pi_n^d$ such that

$$p \in ker(\Lambda). \tag{4}$$

In this paper we introduce and study a multivariate interpolation scheme of Hermite- Birkhoff type. In order to present our results, we need some notions which we present in this section.

For any analytic function $f \in \mathcal{A}_0$, the least term is $f\downarrow = T_j f$, with j the smallest integer for which $T_j f \neq 0$ and $T_j f$ the Taylor polynomial of degree j.

We denote by $p^{[k]}$, the homogeneous component of degree k, of a polynomial. By generalization, for an analytic function at the origin, $f \in \mathcal{A}_0$, the homogeneous component of order k is $f^{[k]} = \sum\limits_{|\alpha|=k} D^k f(0) \cdot x^\alpha/\alpha!$, with $\alpha = (\alpha_1, \ldots, \alpha_d) \in N^d$, $|\alpha| = \alpha_1 + \ldots \alpha_d$, $\alpha! = \alpha_1 \cdot \ldots \cdot \alpha_d$ and $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$.

Some of the properties of the interpolation scheme presented in section 2, are proved using the general interpolation scheme named "least interpolation". This scheme was introduced by C. de Boor and A. Ron. Let Λ be a set of functionals and

$$H_\Lambda = span\{\lambda^\nu;\ \lambda \in \Lambda\}, \tag{5}$$

where λ^ν is the generating function of the functional $\lambda \in \Lambda$.
In [2] is proved that the least space

$$H_\Lambda\downarrow = span\{g\downarrow; g \in H_\Lambda\}, \tag{6}$$

is a minimal interpolation space for the conditions Λ.

The generating function of a functional λ is

$$\lambda^\nu(x) = \sum_{\alpha \in N^d} \frac{D^\alpha \lambda^\nu(0)}{\alpha!} x^\alpha = \sum_{\alpha \in N^d} \frac{\lambda(m_\alpha)}{\alpha!} x^\alpha, \text{ with } m_\alpha(x) = x^\alpha. \tag{7}$$

The generating function can be obtained using the following relation:

$$\lambda^\nu(z) = \lambda(e_z), \text{ with } e_z(x) = e^{z \cdot x}. \tag{8}$$

Very important for the study of the properties of "least interpolation space", is the pair $< f, p >$, between an analytic function f and a polynomial p:

$$< f, p >= (p(D)f)(0) = \sum_{\alpha \in N^d} \frac{D^\alpha p(0) D^\alpha f(0)}{\alpha!}, \tag{9}$$

If $p = \sum\limits_{|\alpha| \leq deg\ p} c_\alpha (\cdot)^\alpha$, then $p(D)$ is the differential operator with constant coefficients:

$$p(D) = \sum_{|\alpha| \leq deg\ p} c_\alpha D^\alpha.$$

Obviously, the pair (9) is a veritable inner product on polynomial spaces.

The action of a functional λ on a polynomial p can be expressed using the generating function of the functional λ and the pair (9):

$$\lambda(p) = < \lambda^\nu, p > = \sum_{\alpha \in N^d} \frac{D^\alpha \lambda^\nu(0) D^\alpha p(0)}{\alpha!} = (p(D)\lambda^\nu)(0).$$

For every $g \in \mathcal{A}_0$ and $p \in \Pi$, the following equalities holds:

$$< g^{[k]}, p > = < g^{[k]}, p^{[k]} >, \tag{10}$$

$$< g, g\downarrow > = < g\downarrow, g\downarrow > . \tag{11}$$

If the functionals from Λ are linear independent, than the generating functions of these functionals represents a basis for the space H_Λ. The following proposition gives an algorithm of obtaining a new basis, g_j, of H_Λ, orthogonal to $g_j\downarrow$, $j \in \{1, \ldots, n\}$.

Proposition 1. ([1]) *If $g_1, \ldots, g_j$ are linear independent elements of the space H_Λ, satisfying the condition $< g_i, g_k\downarrow > \neq 0 \Leftrightarrow k = i$, then the function*

$$g_j = p_j - \sum_{l=1}^{j-1} g_l \frac{< p_j, g_l\downarrow >}{< g_l, g_l\downarrow >},$$

where p_i, $i \in \{1, \ldots, n\}$ is a basis for H_Λ, has the following properties:

1. $< g_j, g_i\downarrow > = 0$, $\forall\ i < j$.
2. $< g_j, g_j\downarrow > \neq 0$.
3. *If deg $g_j\downarrow \leq$ deg $g_i\downarrow$, for one $i < j$, then $< g_i, g_j\downarrow > = 0$.*
4. *If deg $g_j\downarrow >$ deg $g_i\downarrow$, for one $i < j$, then the function*

   $$\tilde{g}_i = g_i - g_j \cdot \frac{< g_i, g_j\downarrow >}{< g_j, g_j\downarrow >},$$

   has the property $< \tilde{g}_i, g_j\downarrow > = 0$.

Using particular choices of the functionals in least interpolation scheme, we can obtain the classical interpolation schemes. The interpolation schemes of Lagrange type are obtained using a set of evaluation functionals on a set of distinct points, $\Theta \subset R^d$, that is $\Lambda = \{\delta_\theta | \theta \in \Theta\}$. The generating functions are given by $\delta_\theta^\nu = e_\theta$. Consequently

$$H_\Lambda = H_\Theta = span\{e_\theta | \theta \in \Theta\}; \quad H_\Lambda\!\downarrow = H_\Theta\!\downarrow = span\{g\!\downarrow |g \in H_\Theta\}. \tag{12}$$

Let be Π^d the space of polynomials in d variables, $\Theta \subset R^d$, a finite set of distinct points and

$$\Lambda = \left\{\lambda_{q,\theta} \left| \lambda_{q,\theta}(p) = (q(D)p)(\theta);\ q \in \mathcal{P}_\theta;\ \theta \in \Theta;\ \mathcal{P}_\theta \subset \Pi^d \right.\right\} \tag{13}$$

The generating function of the functional $\lambda_{q,\theta}$ is

$$\lambda_{q,\theta}^\nu(z) =< \lambda_{q,\theta}, e_z >= q(z)e_\theta(z), \text{ that is } \lambda_{q,\theta}^\nu = q \cdot e_\theta \tag{14}$$

Hence,

$$H_\Lambda = \sum_{\theta \in \Theta} e_\theta \mathcal{P}_\theta \text{ and } H_\Lambda\!\downarrow = \left(\sum_{\theta \in \Theta} e_\theta \mathcal{P}_\theta\right)\!\downarrow$$

This particular choice of the set of functionals, with additional conditions that the polynomial spaces $\mathcal{P}_\theta$ being scalar invariant, is the multidimensional generalization of the univariate interpolation schemes of Hermite-Birkhoff type. If, more, all the spaces $\mathcal{P}_\theta$ are D - invariant, then we obtain a Hermite interpolation scheme.

2. Main results

In this section we consider $d = 2$. We will denote by Π_n^2 the space of all polynomial of degree less or equal n, in two variables, and with Π_n^0 the space of homogeneous polynomials of degree n, in two variables.

Let $\Theta \subset R^2$ be a set of distinct points:

$$\Theta = \{\theta_k = (\xi_k, \eta_k) \mid \Delta_k \neq 0,\ k \in \{0, \ldots, n\}\},\ \text{with} \tag{15}$$

$$\Delta_k = \begin{vmatrix} \xi_0^k & \xi_0^{k-1}\eta_0 & \ldots & \eta_0^k \\ \xi_1^k & \xi_1^{k-1}\eta_1 & \ldots & \eta_1^k \\ \ldots & \ldots & \ldots & \ldots \\ \xi_k^k & \xi_k^{k-1}\eta_k & \ldots & \eta_k^k \end{vmatrix} \neq 0 \tag{16}$$

We consider the following set of functionals:

$$\Lambda_\Theta = \left\{ \lambda_{j,\theta_k} \mid \lambda_{j,\theta_k}(f) = f^{[j]}(\theta_k),\ \theta_k \in \Theta,\ j \in \{0,\dots,n\},\ k \in \{0,\dots j\} \right\} \tag{17}$$

Theorem 1. *Let consider the polynomials:*

$$p_{j,k}(x) = \sum_{|\alpha|=j} \frac{\theta_k^\alpha x^\alpha}{\alpha!},\ \theta_k \in \Theta \tag{18}$$

The polynomial space

$$\mathcal{P} = span\{p_{j,k}(x) \mid j \in \{0,\dots,n\},\ k \in \{0,\dots j\}\}, \tag{19}$$

is a minimal interpolation space for the conditions Λ_Θ, defined in (17).

Proof: The generating function of the functional λ_{j,θ_k} is $\lambda^\nu_{j,\theta_k}(z) = \lambda_{j,\theta_k}(e_z) = e_z^{[j]}(\theta_k) = p_{j,k}(z)$. Using (5) we obtain:

$$H_{\Lambda_\Theta} = span\{\lambda^\nu_{j,\theta_k} | j \in \{0,\dots,n\};\ k \in \{0,\dots,j\}\} \subset \Pi^2. \tag{20}$$

H_{Λ_Θ} is a polynomial space and therefore $H_{\Lambda_\Theta}\downarrow = H_{\Lambda_\Theta} = S$. Consequently S is a minimal interpolation space for Λ_Θ.

Theorem 2. *The set of polynomials $\{p_{j,k} \mid j \in \{0,\dots,n\},\ k \in \{0,\dots j\}\}$ defined in (18), is a basis for the polynomial space $\mathcal{P}$.*

Proof: The polynomials $p_{j,k}$ are generators for the space $\mathcal{P}$. We must only prove that they are linear independent. We will prove that

$$\sum_{j=0}^{n}\sum_{k=0}^{j} a_{j,k} \sum_{|\alpha|=j} \frac{\theta_k^\alpha x^\alpha}{\alpha!} = 0 \ \Rightarrow\ a_{j,k} = 0,\ \forall\ j \in \{0,\dots,n\},\ k \in \{0,\dots j\}. \tag{21}$$

The set of monomials $\{x^\alpha \mid \alpha \in N^2,\ |\alpha| \leq n\}$ represents a basis for the polynomial space Π_n^2. Therefore, the system (21) leads us to $n+1$ homogeneous systems, which determinants are given in (16), $\Delta_k \neq 0$. Hence, all these systems have only the zero solution, that is $a_{j,k} = 0,\ \forall\ j \in \{0,\dots,n\},\ k \in \{0,\dots j\}$.

Corollary 1. *The dimension of the space $\mathcal{P}$ is $N = \frac{(n+1)(n+2)}{2}$.*

We make an indexation of the polynomials $p_{j,k}$:

$$\begin{array}{lcll} P_1 & = & p_{0,0} & \\ P_2 & = & p_{1,0} & \\ P_3 & = & p_{1,1} & \\ \dots & \dots & \dots & \\ P_m & = & p_{j,0} & \text{with } m = \frac{j(j+1)}{2}+1 \\ P_{m+1} & = & p_{j,1} & \\ P_{m+j} & = & p_{j,j} & \\ \dots & \dots & \dots & \\ P_N & = & p_{n,n} & \end{array} \tag{22}$$

If $i \in \{m, \dots, m+j\}$ then the polynomials P_j are homogeneous polynomials of degree j.

Corollary 2. *Let P_j be the polynomials defined in (22). The orthogonalization algorithm from the proposition becomes the classical Gramm- Schmidt orthogonalization algorithm with respect to the inner product defined in (9). More, if $m = \frac{j(j+1)}{2}+1$, then $g_m = P_m = p_{j,0}$ and $g_l \in \Pi_j^0$, $\forall\, l \in \{m, \dots, m+j\}$.*

Theorem 3. *The space Π_n^2 is an interpolation space for the conditions Λ_Θ.*

Proof: We easily observe that $card\ \Lambda_\Theta = dim\ \Pi_n^2 = \begin{pmatrix} n+2 \\ 2 \end{pmatrix}$. If there is not any polynomial $p \in \Pi_n^2$ such that $p \in ker(\Lambda_\Theta)$, then Π_n^2 is an interpolation space for Λ_Θ. We look for $ker(\Lambda_\Theta)$.

$$ker(\Lambda_\Theta) = \{p \in \Pi \mid \lambda_{j,k}(p) = 0,\ \forall\ j \in \{0, \dots, n\},\ k \in \{0, \dots, j\}\}$$

Let be $p = \sum_{|\alpha| \le n} c_\alpha x^\alpha \in \Pi_n^2$. Then, $p \in ker(\Lambda_\Theta)$ if and only if $p^{[j]}(\theta_k) = 0$, $\forall\ j \in \{0, \dots, n\}$, $k \in \{0, \dots, j\}$. The coefficients of the homogeneous component $p^{[k]}$ are obtained from an homogeneous system with the determinant, $\Delta_k \neq 0$. Consequently, $c_\alpha = 0$, $\forall \alpha <= n$, that is $ker(\Lambda_\Theta) = \{0\}$.

Corollary 3. *The orthogonal basis g_j of the space $\mathcal{P}$, given in the proposition is the monomial basis of the polynomial space Π_n^2.*

Theorem 4. *The interpolation operator with respect to the conditions* Λ_Θ *is the Taylor operator of degree* n*, that is*

$$L_{\Lambda_\Theta}(f) = T_n(f) = \sum_{|\alpha|\leq n} \frac{D^\alpha f(0) x^\alpha}{\alpha!} \tag{23}$$

Proof: We prove that $\lambda_{j,\theta_k}(f) = \lambda_{j,\theta_k}(L_{\Lambda_\Theta}(f)) = \sum_{|\alpha|=j} \frac{\theta_k^\alpha D^\alpha f(0)}{\alpha!}$

$$\lambda_{j,\theta_k}(L_{\Lambda_\Theta}(f) = \left\langle \lambda^\nu_{j,\theta_k}, L_{\Lambda_\Theta} \right\rangle = \sum_{|\alpha|=j} \sum_{|\beta|\leq n} \frac{\theta_k^\alpha}{\alpha!} \cdot \frac{D^\beta f(0)}{\beta!} \left\langle x^\alpha, x^\beta \right\rangle = \sum_{|\alpha|=j} \frac{\theta_k^\alpha D^\alpha f(0)}{\alpha!}$$

Definition 1. *Let* $L : X \to Y$ *be a linear operator. The dual operator* L^* *is the operator* $L^* : Y' \to X'$ *having the property:*

$$< L^*(g), f >=< g, L(f) >, \;\; g \in Y', \;\; f \in X, \tag{24}$$

We use the identification between the algebraic dual, Π' of the space of multivariate polynomials with the space $R[[X]]$ of formal power series, given in [2]. In the same paper is proved that a polynomial can be consider as an element of Π, as a linear functional (power series) in Π' and as an analytic function on $R^\#$. Furthermore, many non-polynomial $\lambda \in \Pi'$ of interest, can also be reasonably interpreted as a function analytic at origin. Taking into account the previous specifications we can formulate the following theorem:

Theorem 5. *The dual operator* $L^* : \mathcal{A}_0 \to \Pi$ *is given by:*

$$L^*_{\Lambda_\Theta}(f) = \sum_{|\alpha|\leq n} \frac{D^\alpha f(0) x^\alpha}{\alpha!} \tag{25}$$

Proof: By a simple calculation we obtain:

$$< L_{\Lambda_\Theta}(f), q >=< f, L^*_{\Lambda_\Theta}(q) >= \sum_{|\alpha|\leq n} \frac{D^\alpha f(0) \cdot D^\alpha q(0)}{\alpha!}$$

Corollary 4. *The operators L_{Λ_Θ} and $L^*_{\Lambda_\Theta}$ coincide.*

The following proposition proves that the interpolation scheme $(\Lambda_\Theta, \Pi_n^2)$ is a Hermite-Birkhoff interpolation scheme, in the sense formulated in (13).

Proposition 2. *The interpolation scheme $(\Lambda_\Theta, \Pi_n^2)$ is equivalent with the interpolation scheme $(\tilde{\Lambda}_\Theta, \Pi_n^2)$, with*

$$\tilde{\Lambda}_\Theta = \left\{ \lambda_{q_j,\theta_k} \mid \lambda_{q_j,\theta_k} = (q_j(D)p)(\theta_k),\ q_j \in \mathcal{P}_{\theta_k^{[j]}};\ k \in \{0, \dots n\} \right\}, \tag{26}$$

$$\mathcal{P}_{\theta_k^{[j]}} = \left\{ q \in \Pi^2 \mid q(x) = \sum_{|\alpha|=j} \frac{\theta_k^\alpha x^\alpha}{\alpha!} + q_1(x),\ deg(q_1) > j \right\} \tag{27}$$

Proof: The point θ_k intervenes in $n-k+1$ conditions. That is why, we consider that in the set Θ we have $n-k+1$ copies of θ_k, which we will denote by $\theta_k^{[j]}$, $j \in \{k, \dots n\}$. With this notation, the polynomials in the space $\mathcal{P}_{\theta_k^{[j]}}$ satisfy the equality:

$$\left(q(z) \cdot e^{\theta_k \cdot z}\right)\downarrow = \left(e^{\theta_k \cdot z}\right)^{[j]} \tag{28}$$

Hence the interpolation scheme $(\Lambda_\Theta, \Pi_n^2)$ and $(\tilde{\Lambda}_\Theta, \Pi_n^2)$ are equivalent. Obviously, the spaces $\mathcal{P}_{\theta_k^{[j]}}$ are scalar invariant. Consequently, the interpolation scheme $(\Lambda_\Theta, \Pi_n^2)$ is one of Hermite - Birkhoff type.♠

The conditions in the set $(\tilde{\Lambda}_\Theta, \Pi_n^2)$ are not linear independent. The next proposition improves the set of conditions.

Proposition 3. *The interpolation scheme $(\Lambda_\Theta, \Pi_n^2)$ is equivalent with the interpolation scheme $(\overline{\Lambda}_\Theta, \Pi_n^2)$, with*

$$\overline{\Lambda}_\Theta = \left\{ \lambda_{q,\theta_k} \mid \lambda_{q,\theta_k} = (q(D)p)(\theta_k),\ q \in \mathcal{P}_{\theta_k};\ k \in \{0, \dots n\} \right\}, \tag{29}$$

$$\mathcal{P}_{\theta_k} = span \left\{ p_m \;\middle|\; p_m(x) = \begin{cases} \sum\limits_{|\alpha|=m} \dfrac{\theta_k^\alpha x^\alpha}{\alpha!} & , m \in \{k, \dots n\} \\ p_m = 0 & , m < k \end{cases} \right\} \tag{30}$$

Proof: We easily observe that

$$span \left\{ \lambda_{q_j,\theta_k} \mid q_j \in \mathcal{P}_{\theta_k^{[j]}};\ k \in \{0, \dots n\} \right\} = span \left\{ \lambda_{q,\theta_k} \mid q \in \mathcal{P}_{\theta_k};\ k \in \{0, \dots n\} \right\}$$

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