SOME PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Defining the subclasses $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ of certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, some properties for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ are discussed. In this present paper, some coefficient estimates and some interesting applications of Jack's lemma for functions $f(z)$ in the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ are given.

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1.Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Shams, Kulkarni and Jahangiri [3] have considered the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of $f(z)$ which satisfy

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in \mathbb{U})$$

for some $\alpha(\alpha \geq 0)$ and $\beta(0 \leq \beta < 1)$. The class $\mathcal{KD}(\alpha,\beta)$ is defined by the subclass of $\mathcal{A}$ consisting of $f(z)$ such that $zf'(z) \in \mathcal{SD}(\alpha,\beta)$. In view of the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$, we introduce the subclass $\mathcal{MD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in \mathbb{U})$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$. The class $\mathcal{ND}(\alpha,\beta)$ is also defined by $f(z) \in \mathcal{ND}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha,\beta)$. The classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$ were introduced by Nishiwaki and Owa [2]. We discuss some properties of functions $f(z)$ belonging to the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$. We note if $f(z) \in \mathcal{MD}(\alpha,\beta)$, then $\dfrac{zf'(z)}{f(z)} = u + iv$ maps $\mathbb{U}$ onto the elliptic domain such that

$$\left(u - \frac{\alpha^2-\beta}{\alpha^2-1}\right)^2 + \frac{\alpha^2}{\alpha^2-1}v^2 < \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta-1)}v^2 + \frac{\beta+1}{2}$$

for $\alpha = -1$, and the hyperbolic domain such that

$$\left(u - \frac{\alpha^2-\beta}{\alpha^2-1}\right)^2 - \frac{\alpha^2}{1-\alpha^2}v^2 > \frac{\alpha^2(\beta-1)^2}{(\alpha^2-1)^2}$$

for $-1 < \alpha < 0$.

2. Coefficient estimates for the classes $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$

By definitions of $\mathcal{MD}(\alpha,\beta)$ and $\mathcal{ND}(\alpha,\beta)$, we derive

Theorem 1. *If $f(z) \in \mathcal{MD}(\alpha,\beta)$, then*

$$f(z) \in \mathcal{MD}\left(0, \frac{\beta-\alpha}{1-\alpha}\right).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha,\beta)$,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \leq \alpha\,\mathrm{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta \qquad (z \in \mathbb{U})$$

implies that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \frac{\beta - \alpha}{1 - \alpha} \qquad (\alpha \leq 0, \beta > 1).$$

Since $\dfrac{\beta - \alpha}{1 - \alpha} > 1$, we prove the theorem.

Corollary 1. *If $f(z) \in \mathcal{ND}(\alpha, \beta)$, then*

$$f(z) \in \mathcal{ND}\left(0, \frac{\beta - \alpha}{1 - \alpha}\right).$$

Our first result for the coefficient estimates of $f(z)$ in the class $\mathcal{MD}(\alpha, \beta)$ is contained in

Theorem 2. *If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then*

$$|a_2| \leq \frac{2(\beta - 1)}{1 - \alpha}$$

and

$$|a_n| \leq \frac{2(\beta - 1)}{(n-1)(1-\alpha)} \prod_{j=1}^{n-2}\left(1 + \frac{2(\beta - 1)}{j(1-\alpha)}\right) \qquad (n \geq 3).$$

Proof. If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then

$$\beta - \alpha + (\alpha - 1)\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$$

from Theorem 1. And let us define the function $p(z)$ by

$$p(z) = \frac{\beta - \alpha + (\alpha - 1)\dfrac{zf'(z)}{f(z)}}{\beta - 1}. \qquad (1)$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\operatorname{Re} p(z)>0$ $(z\in\mathbb{U})$. Therefore, if we write

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n, \tag{2}$$

then $|p_n|\leq 2$ $(n\geq 1)$. From (1) and (2), we obtain that

$$(\alpha-1)\sum_{n=2}^{\infty}(n-1)a_nz^n=(\beta-1)\sum_{n=1}^{\infty}p_nz^n(z+\sum_{n=2}^{\infty}a_nz^n).$$

Therefore we have

$$a_n=\frac{\beta-1}{(n-1)(\alpha-1)}(p_{n-1}+p_{n-2}a_2+\cdots+p_2a_{n-2}+p_1a_{n-1})$$

for all $n\geq 2$. When $n=2$,

$$|a_2|\leq\frac{\beta-1}{1-\alpha}|p_1|\leq\frac{2(\beta-1)}{1-\alpha}.$$

And when $n=3$,

$$\begin{aligned}|a_3|&\leq\frac{\beta-1}{2(1-\alpha)}(|p_2|+|p_1||a_2|)\\&\leq\frac{2(\beta-1)}{2(1-\alpha)}\left(1+\frac{2(\beta-1)}{1-\alpha}\right).\end{aligned}$$

Let us suppose that

$$|a_k|\leq\frac{2(\beta-1)}{(k-1)(1-\alpha)}(1+|a_2|+\cdots+|a_{k-2}|+|a_{k-1}|) \tag{3}$$

$$\leq\frac{2(\beta-1)}{(k-1)(1-\alpha)}\prod_{j=1}^{k-2}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right)\qquad(k\geq 3).$$

Then we see

$$1+|a_2|+\cdots+|a_{k-2}|+|a_{k-1}|\leq\prod_{j=1}^{k-2}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right). \tag{4}$$

By using (3) and (4), we obtain that

$$\begin{aligned}|a_{k+1}| &\leq \frac{2(\beta-1)}{k(1-\alpha)}(1+|a_2|+\cdots+|a_{k-2}|+|a_{k-1}|+|a_k|)\\ &\leq \left(1+\frac{2(\beta-1)}{(k-1)(1-\alpha)}\right)\frac{2(\beta-1)}{k(1-\alpha)}\prod_{j=1}^{k-2}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right)\\ &\leq \frac{2(\beta-1)}{k(1-\alpha)}\prod_{j=1}^{k-1}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right).\end{aligned}$$

This completes the proof of the Theorem.

COROLLARY 2.*If* $f(z) \in \mathcal{ND}(\alpha, \beta)$, *then*

$$|a_2| \leq \frac{2(\beta-1)}{2(1-\alpha)}$$

and

$$|a_n| \leq \frac{2(\beta-1)}{n(n-1)(1-\alpha)}\prod_{j=1}^{n-2}\left(1+\frac{2(\beta-1)}{j(1-\alpha)}\right) \qquad (n \geq 3).$$

Proof. From $f(z) \in \mathcal{ND}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha, \beta)$, replacing a_n by na_n in Theorem 2, we have the corollary.

3.Applications of Jack's lemma for the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$

In this section, some applications of Jack's lemma for $f(z)$ belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ are discussed. Next lemma was given by Jack [1].

LEMMA 1. *Let the function* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$. *If*

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|,$$

then

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Applying the above lemma, we derive

THEOREM 3. *If $f(z) \in \mathcal{MD}(\alpha, \beta)$, then*

$$\left| \left(\frac{f(z)}{z} \right)^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \qquad (\delta \geq 0)$$

for some $\alpha (\alpha \leq 0)$ and $\beta (\beta > 1)$, or

$$\left| \left(\frac{f(z)}{z} \right)^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}} - 1 \right| < 1 + \delta \qquad (\delta \geq 0)$$

for some $\alpha (\alpha \leq -1)$ and $\beta (\beta > 1)$.

Proof. Let us define

$$\gamma = \frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)} > 0,$$

for $\alpha \leq 0$ and $\beta > 1$, and

$$\gamma = \frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)} < 0$$

for $\alpha \leq -1$ and $\beta > 1$. Further, let the funcion $w(z)$ be defined by

$$w(z) = \frac{\left(\frac{f(z)}{z} \right)^{\gamma} - 1}{1 + \delta} \qquad (\delta \geq 0)$$

which is equivalent to

$$\frac{zf'(z)}{f(z)} - 1 = \frac{(1+\delta) z w'(z)}{\gamma \{(1+\delta) w(z) + 1\}}.$$

Then we see that $w(z)$ is analytic in $\mathbb{U}$, and $w(0) = 0$. On the other hand, if $f(z) \in \mathcal{MD}(\alpha, \beta)$ $(\alpha \leq 0, \beta > 1)$, then

$$\text{Re}\left(\frac{zf'(z)}{f(z)} \right) - \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| = 1 + \frac{1}{\gamma} \text{Re}\left(\frac{(1+\delta) z w'(z)}{(1+\delta) w(z) + 1} \right) - \frac{\alpha}{|\gamma|} \left| \frac{(1+\delta) z w'(z)}{(1+\delta) w(z) + 1} \right| < \beta.$$

Furthermore, if there is a point z_0 $(z_0 \in \mathbb{U})$ which satisfies

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1,$$

then Lemma 1 gives us that

$$1+\frac{1}{\gamma}\mathrm{Re}\left(\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right)-\frac{\alpha}{|\gamma|}\left|\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right|$$
$$=1+\frac{k(1+\delta)}{\gamma}\mathrm{Re}\left(\frac{1}{(1+\delta)+e^{-i\theta}}\right)-\frac{\alpha k(1+\delta)}{|\gamma|}\left|\frac{1}{(1+\delta)+e^{-i\theta}}\right|$$
$$=1+\frac{k(1+\delta)}{\gamma}\cdot\frac{1+\delta+cos\theta-\alpha\sqrt{(1+\delta)^2+2(1+\delta)cos\theta+1}}{(1+\delta)^2+2(1+\delta)cos\theta+1}=F(\theta).$$

When $\gamma > 0$,

$$F(\theta)\geq 1+\frac{k(1+\delta)(1-\alpha)}{\gamma(2+\delta)}$$
$$\geq 1+\frac{(1+\delta)(1-\alpha)}{\gamma(2+\delta)}=\beta,$$

because

$$\gamma=\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}.$$

Further, when $\gamma < 0$,

$$F(\theta)\geq 1+\frac{k(1+\delta)(1+\alpha)}{\gamma(2+\delta)}$$
$$\geq 1+\frac{(1+\delta)(1+\alpha)}{\gamma(2+\delta)}=\beta.$$

This contradicts our condition of the theorem. Thus there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. This completes the proof of the Theorem.

Corollary 3. *If $f(z) \in \mathcal{ND}(\alpha,\beta)$, then*

$$\left|(f'(z))^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \qquad (\delta\geq 0)$$

for some $\alpha(\alpha \leq 0)$ and $\beta(\beta > 1)$, or

$$\left|(f'(z))^{\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \qquad (\delta\geq 0)$$

for some $\alpha(\alpha \leq -1)$ *and* $\beta(\beta > 1)$.

Proof. Replacing $f(z)$ by $zf'(z)$ in Theorem 3, we have the corollary 3.

References

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