NOTE ON CERTAIN ANALYTIC FUNCTIONS

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Abstract.Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{A}$, a subclass $\mathcal{B}_k(\alpha,\beta,\gamma)$ of $\mathcal{A}$ is introduced. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$.

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1.Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be a member of the subclass $\mathcal{B}_k(\alpha,\beta,\gamma)$ of $\mathcal{A}$ if it satisfies

$$\mathrm{Re}\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\} > \gamma \qquad (k \in \mathbb{N} = \{1,2,3,\dots\}\,;\, z \in \mathbb{U}) \tag{2}$$

for some $a_j \in \mathbb{R}$ $(j = 2,3,4,\dots,k), \alpha \in \mathbb{R}, \beta \in \mathbb{R}$ $(\beta \neq 0)$, and $\gamma \in \mathbb{R}$ $(0 \leq \gamma < k!\alpha a_k\,;\, a_1 = 1)$. We consider some properties for functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$.

Remark 1. $\mathcal{B}_k(\alpha,\beta,\gamma)$ is convex.

Because, for $f(z) \in \mathcal{B}_k(\alpha,\beta,\gamma)$ and $g(z) \in \mathcal{B}_k(\alpha,\beta,\gamma)$, we define

$$F(z) = (1-t)f(z) + tg(z) \qquad (0 \leq t \leq 1).$$

Then

$$\begin{aligned}
&\mathrm{Re}\left\{\alpha F^{(k)}(z) + \beta F^{(k+1)}(z)\right\} \\
&= \mathrm{Re}\left\{\alpha(1-t)f^{(k)}(z) + \alpha t g^{(k)}(z) + \beta(1-t)zf^{(k+1)}(z) + \beta t z g^{(k+1)}(z)\right\} \\
&= (1-t)\mathrm{Re}\left\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\right\} + t\mathrm{Re}\left\{\alpha g^{(k)}(z) + \beta z g^{(k+1)}(z)\right\} \\
&> (1-t)\gamma + t\gamma = \gamma.
\end{aligned}$$

Therefore $F(z) \in \mathcal{B}_k(\alpha, \beta, \gamma)$, that is, $\mathcal{B}_k(\alpha, \beta, \gamma)$ is convex.

In the present paper, we consider some properties of functions $f(z)$ belonging to the class $\mathcal{B}_k(\alpha, \beta, \gamma)$.

2.Properties of the class $\mathcal{B}_1(\alpha, \beta, \gamma)$ and $\mathcal{B}_2(\alpha, \beta, \gamma)$

We begin with the statement and the proof of the following result.
For cases $k = 1$, we obtain

Theorem 1. *A function $f(z) \in \mathcal{A}$ is in the class of $\mathcal{B}_1(\alpha, \beta, \gamma)$ if and only if*

$$f(z) = z + 2(\alpha - \gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{1}{n((n-1)\beta + \alpha)}x^{n-1}z^n\right)d\mu(x) \qquad (3)$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

Proof. For $f(z) \in \mathcal{A}$, we define

$$p(z) = \frac{\alpha f'(z) + \beta z f''(z) - \gamma}{\alpha - \gamma}. \qquad (4)$$

Then $p(z)$ is Carathéodory function . Therefore we can write

$$\frac{\alpha f'(z)+\beta z f''(z)-\gamma}{\alpha-\gamma}=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x) \qquad \text{(see[1])}. \tag{5}$$

It follows from (5) that

$$z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f'(z)+zf''(z)\right)=\frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(\alpha-\gamma)\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x)\right\} \tag{6}$$
$$=\frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(\alpha-\gamma)\int_{|x|=1}(1+xz)(1+xz+x^2z^2+\dots)d\mu(x)\right\}.$$

Integrating the both sides of (6), we know that

$$\int_0^z \zeta^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f'(\zeta)+\zeta f''(\zeta)\right)d\zeta=$$

$$=\frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z\left(\alpha\zeta^{\frac{\alpha}{\beta}-1}+2(\alpha-\gamma)\sum_{n=1}^{\infty}x^n\zeta^{n+\frac{\alpha}{\beta}-1}\right)d\zeta\right\}d\mu(x),$$

that is, that

$$\begin{aligned} z^{\frac{\alpha}{\beta}}f'(z) &= \frac{1}{\beta}\int_{|x|=1}\left\{\beta z^{\frac{\alpha}{\beta}}+2(\alpha-\gamma)\left(\sum_{n=1}^{\infty}\frac{\beta}{n\beta+\alpha}x^nz^{n+\frac{\alpha}{\beta}}\right)\right\}d\mu(x) \\ &= z^{\frac{\alpha}{\beta}}+2(\alpha-\gamma)z^{\frac{\alpha}{\beta}}\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^nz^n\right)d\mu(x). \end{aligned}$$

Thus, we have

$$f'(z)=1+2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^nz^n\right)d\mu(x). \tag{7}$$

An integration of both sides in (7) gives us that

$$\int_0^z f'(\zeta)d\zeta=\int_0^z\left\{1+2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{n\beta+\alpha}x^n\zeta^n\right)d\mu(x)\right\}d\zeta,$$

or

$$
\begin{aligned}
f(z) &= z + 2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n\beta+\alpha)}x^n z^{n+1}\right)d\mu(x) \\
&= z + 2(\alpha-\gamma)\int_{|x|=1}\left(\sum_{n=2}^{\infty}\frac{1}{n\left((n-1)\beta+\alpha\right)}x^{n-1} z^{n}\right)d\mu(x).
\end{aligned}
$$

This completes the proof of Theorem 1.

Corollary 1. *The extreme points of* $\mathcal{B}_1(\alpha,\beta,\gamma)$ *are*

$$f_x(z) = z + 2(\alpha-\gamma)\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}z^n \qquad (|x|=1).$$

In view of Theorem 1, we have the following corollary for a_n.

Corollary 2. *If* $f(z)\in\mathcal{A}$ *is in the class* $\mathcal{B}_1(\alpha,\beta,\gamma)$*, then*

$$|a_n| \leqq \frac{2(\alpha-\gamma)}{n\left((n-1)\beta+\alpha\right)} \qquad (n=2,3,4,\dots).$$

Equality holds for the function $f(z)$ *given by*

$$f(z) = z + 2(\alpha-\gamma)\sum_{n=2}^{\infty}\frac{x^{n-1}}{n\left((n-1)\beta+\alpha\right)}z^n \qquad (|x|=1).$$

Further, the following distortion inequality follows from Theorem 1.

Corollary 3. *If* $f(z)\in\mathcal{A}$ *is in the class* $\mathcal{B}_1(\alpha,\beta,\gamma)$*, then*

$$|f(z)| \leqq |z| + 2(\alpha-\gamma)\left(\sum_{n=2}^{\infty}\frac{|z|^n}{n\left((n-1)\beta+\alpha\right)}\right) \qquad (z\in\mathbb{U}).$$

Remark 2. If $\beta > 0$ and $\dfrac{\alpha}{\beta} = j$ $(j = 2, 3, 4, \dots)$ in Corollary 3, then we see that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{|z|^n}{n\left((n-1)\beta+\alpha\right)} &\leq \frac{|z|^2}{\beta} \sum_{n=2}^{\infty} \frac{1}{n(n+j-1)} \\ &= \frac{|z|^2}{\beta(j-1)} \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+j-1}\right) \\ &= \frac{|z|^2}{\beta(j-1)} \sum_{n=2}^{j} \frac{1}{n} < \frac{\log(j)}{\beta(j-1)} |z|^2.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}|f(z)| &< |z| + \frac{2(\alpha-\gamma)\log(j)}{\beta(j-1)} |z|^2 \\ &< 1 + \frac{2(\alpha-\gamma)\log(j)}{\beta(j-1)}.\end{aligned}$$

Next, for cases $k = 2$ we show

Theorem 2. *A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{B}_2(\alpha, \beta, \gamma)$ if and only if*

$$f(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)\left((n-2)\beta+\alpha\right)} z^n \right) d\mu(x)$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

Proof. For $f(z) \in \mathcal{A}$, we define

$$p(z) = \frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma}.$$

Then $p(z)$ is Carathéodory function. Hence, we can write

$$\frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x). \tag{8}$$

In view of (8), we have that

$$
\begin{aligned}
z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f''(z)+zf'''(z)\right) &= \\
\frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(2\alpha a_2-\gamma)\int_{|x|=1}\left(1+2\sum_{n=1}^{\infty}x^n z^n\right)d\mu(x)\right\} \\
=\frac{1}{\beta}\int_{|x|=1}\left(2\alpha a_2 z^{\frac{\alpha}{\beta}-1}+2(2\alpha a_2-\gamma)\sum_{n=1}^{\infty}x^n z^{n+\frac{\alpha}{\beta}-1}\right)d\mu(x).
\end{aligned} \tag{9}
$$

Integrating the both sides of (9), we have that

$$
\begin{aligned}
&\int_0^z \zeta^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f''(\zeta)+\zeta f'''(\zeta)\right)d\zeta \\
&=\frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z\left(2\alpha a_2\zeta^{\frac{\alpha}{\beta}-1}+2(2\alpha a_2-\gamma)\left(\sum_{n=1}^{\infty}x^n\zeta^{n+\frac{\alpha}{\beta}-1}\right)\right)d\zeta\right\}d\mu(x),
\end{aligned}
$$

that is, that

$$
z^{\frac{\alpha}{\beta}}f''(z)=\frac{1}{\beta}\int_{|x|=1}\left\{2\beta a_2 z^{\frac{\alpha}{\beta}}+2(2\alpha a_2-\gamma)\left(\sum_{n=1}^{\infty}\frac{\beta}{n\beta+\alpha}x^n z^{n+\frac{\alpha}{\beta}}\right)\right\}d\mu(x).
$$

This implies that

$$
f''(z)=\int_{|x|=1}\left\{2a_2+2(2\alpha a_2-\gamma)\left(\sum_{n=1}^{\infty}\frac{x^n}{n\beta+\alpha}z^n\right)\right\}d\mu(x). \tag{10}
$$

An integration of both sides in (10) gives us that

$$
\int_0^z f''(\zeta)d\zeta=\int_0^z\left\{2a_2+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{x^n}{n\beta+\alpha}\zeta^n\right)d\mu(x)\right\}d\zeta
$$

or

$$
f'(z)-1=2a_2 z+2(2\alpha a_2-\gamma)\int_{|x|=1}\left(\sum_{n=1}^{\infty}\frac{x^n}{(n+1)(n\beta+\alpha)}z^{n+1}\right)d\mu(x).
$$

Therefore, we know that

$$f'(z) = 1 + 2a_2 z + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=2}^{\infty} \frac{x^{n-1}}{n\left((n-1)\beta + \alpha\right)} z^n \right) d\mu(x). \quad (11)$$

Applying the same method for (11), we see that

$$\int_0^z f'(\zeta) d\zeta =$$

$$= \int_0^z \left\{ 1 + 2a_2 \zeta + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=2}^{\infty} \frac{x^{n-1}}{n\left((n-1)\beta + \alpha\right)} \zeta^n \right) d\mu(x) \right\} d\zeta.$$

Thus, we obtain that

$$\begin{aligned} f(z) &= z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=2}^{\infty} \frac{x^{n-1}}{(n+1)n\left((n-1)\beta + \alpha\right)} z^{n+1} \right) d\mu(x) \\ &= z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left(\sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)\left((n-2)\beta + \alpha\right)} z^n \right) d\mu(x) \end{aligned}$$

This completes the proof of Theorem 2.

Corollary 4. *The extreme points of* $\mathcal{B}_2(\alpha, \beta, \gamma)$ *are*

$$f_x(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \left(\sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)\left((n-2)\beta + \alpha\right)} z^n \right) \qquad (|x| = 1).$$

In view of Theorem 2, we have the following corollary for a_n.

Corollary 5. *If* $f(z) \in \mathcal{A}$ *is in the class* $\mathcal{B}_2(\alpha, \beta, \gamma)$, *then*

$$|a_n| \leq \frac{2(2\alpha a_2 - \gamma)}{n(n-1)\left((n-2)\beta + \alpha\right)} \qquad (n = 3, 4, 5, \dots).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z+a_2z^2+2(2\alpha a_2-\gamma)\left(\sum_{n=3}^{\infty}\frac{x^{n-2}}{n(n-1)\left((n-2)\beta+\alpha\right)}z^n\right) \qquad (|x|=1).$$

Further, the following distortion inequality follows from Theorem 2.

Corollary 6. *If $f(z) \in \mathcal{A}$ is in the class $\mathcal{B}_2(\alpha,\beta,\gamma)$, then*

$$|f(z)| \leq |z|+|a_2||z|^2+2(2\alpha a_2-\gamma)\left(\sum_{n=3}^{\infty}\frac{|z|^n}{n(n-1)\left((n-2)\beta+\alpha\right)}\right) \qquad (z \in \mathbb{U}).$$

3.Properties of the class $\mathcal{B}_k(\alpha,\beta,\gamma)$

For cases k is any natural number, we have

Theorem 3.*A function $f(z) \in \mathcal{A}$ belongs to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$ if and only if*

$$f(z) = z + a_2z^2 + \dots$$

$$+a_kz^k+2(k!\alpha a_k-\gamma)\int_{|x|=1}\left(\sum_{n=k+1}^{\infty}\frac{x^{n-k}z^n}{n(n-1)\dots(n-k+1)\left((n-k)\beta+\alpha\right)}\right)d\mu(x)$$

for $k = 1,2,3,\dots$, where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : |x| = 1\}$.

*Proof.*For $f(z) \in \mathcal{A}$, we define

$$p(z) = \frac{\alpha f^{(k)}(z)+\beta z f^{(k+1)}(z)-\gamma}{k!\alpha a_k-\gamma}.$$

Since $p(z)$ is Carathéodory function, we can write that

$$\frac{\alpha f^{(k)}(z)+\beta z f^{(k+1)}(z)-\gamma}{k!\alpha a_k-\gamma} = \int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x). \qquad (12)$$

This means that

$$z^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f^{(k)}(z)+zf^{(k+1)}(z)\right)=$$

$$\frac{1}{\beta}z^{\frac{\alpha}{\beta}-1}\left\{\gamma+(k!\alpha a_k-\gamma)\int_{|x|=1}\left(1+2\sum_{n=1}^{\infty}x^n z^n\right)d\mu(x)\right\}= \tag{13}$$

$$\frac{1}{\beta}\int_{|x|=1}\left(k!\alpha a_k z^{\frac{\alpha}{\beta}-1}+2(k!\alpha a_k-\gamma)\sum_{n=1}^{\infty}x^n z^{n+\frac{\alpha}{\beta}-1}\right)d\mu(x).$$

Integrating the both sides of (13), we obtain that

$$\int_0^z \zeta^{\frac{\alpha}{\beta}-1}\left(\frac{\alpha}{\beta}f^{(k)}(\zeta)+\zeta f^{(k+1)}(\zeta)\right)d\zeta$$

$$=\frac{1}{\beta}\int_{|x|=1}\left\{\int_0^z\left(k!\alpha a_k\zeta^{\frac{\alpha}{\beta}-1}+2(k!\alpha a_k-\gamma)\left(\sum_{n=1}^{\infty}x^n\zeta^{n+\frac{\alpha}{\beta}-1}\right)\right)d\zeta\right\}d\mu(x),$$

that is, that

$$z^{\frac{\alpha}{\beta}}f^{(k)}(z)=\frac{1}{\beta}\int_{|x|=1}\left\{k!\beta a_k z^{\frac{\alpha}{\beta}}+2(k!\alpha a_k-\gamma)\left(\sum_{n=1}^{\infty}\frac{\beta}{n\beta+\alpha}x^n z^{\frac{\alpha}{\beta}-1}\right)\right\}d\mu(x).$$

This is equivalent to

$$f^{(k)}(z)=\int_{|x|=1}\left\{k!a_k+2(k!\alpha a_k-\gamma)\left(\sum_{n=1}^{\infty}\frac{x^n}{n\beta+\alpha}z^n\right)\right\}d\mu(x). \tag{14}$$

Now, since $f(0)=0$, $f'(0)=1$, and $f^{(m)}(0)=m!a_m$ $(m=2,3,4,\dots)$, we see that

$$\begin{aligned}\int_0^z f^{(m)}(\zeta)d\zeta &= f^{(m-1)}(z)-f^{(m-1)}(0)\\ &= f^{(m-1)}(z)-(m-1)!a_{m-1}.\end{aligned}$$

Furthermore, we know that

$$\int_0^z \int_0^{\zeta_m} \dots \int_0^{\zeta_2} m! a_m d\zeta_1 d\zeta_2 \dots d\zeta_m = a_m z^m,$$

and

$$\sum_{n=1}^{\infty} \frac{x^n z^{n+k}}{(n+k)(n+k-1)\dots(n+1)(n\beta+\alpha)} =$$

$$\sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)\left((n-k)\beta+\alpha\right)}.$$

Therefore, integrating k times the both sides in (14), we obtain that

$$\int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} f^{(k)}(\zeta_1) d\zeta_1 d\zeta_2 \dots d\zeta_k$$

$$= \int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} \left\{ k! a_k + 2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta+\alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k,$$

that is, that

$$f(z) = f(0) + \int_0^z f'(0) d\zeta_1 + \int_0^z \int_0^{\zeta_2} f''(0) d\zeta_1 d\zeta_2 + \int_0^z \int_0^{\zeta_3} \int_0^{\zeta_2} f'''(0) d\zeta_1 d\zeta_2 d\zeta_3 + \dots$$

$$+ \int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} \left\{ k! a_k + 2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta+\alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k.$$

Thus, we conclude that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots + a_k z^k$$

$$+2(k!\alpha a_k - \gamma) \int_{|x|=1} \left(\sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)\left((n-k)\beta+\alpha\right)} \right) d\mu(x).$$

The proof of Theorem 3 is complete.

Corollary 7. *The extreme points of* $\mathcal{B}_k(\alpha,\beta,\gamma)$ *are*

$$f_x(z) = z + a_2z^2 + a_3z^3 + \cdots + a_kz^k$$

$$+2(k!\alpha a_k - \gamma)\left(\sum_{n=k+1}^{\infty} \frac{x^{n-k}z^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right) \qquad (|x|=1).$$

In view of Theorem 3, we see that

Corollary 8.*If $f(z)$ belongs to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$, then*

$$|a_n| \leq \frac{2(k!\alpha a_k - \gamma)}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)} \qquad (n = k+1, k+2, k+3, \ldots).$$

Equality holds for the function $f(z)$ given by

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_kz^k$$

$$+2(k!\alpha a_k - \gamma)\left(\sum_{n=k+1}^{\infty} \frac{x^{n-k}z^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right) \qquad (|x|=1).$$

Further, the following distortion inequality follows from Theorem 3.

Corollary 9.*If $f(z)$ belongs to the class $\mathcal{B}_k(\alpha,\beta,\gamma)$, then*

$$|f(z)| \leq |z| + |a_2||z|^2 + |a_3||z|^3 + \cdots + |a_k||z|^k$$

$$+2(k!\alpha a_k - \gamma)\left(\sum_{n=k+1}^{\infty} \frac{|z|^n}{n(n-1)\ldots(n-k+1)\left((n-k)\beta+\alpha\right)}\right) \qquad (z \in \mathbb{U}).$$

References

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