# ON A CLASS OF MODULAR NUMERICAL SEMIGROUPS

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Abstract. In this study, we give some results about $S = S(a, a^2)$ modular numerical semigroups, where $a \geq 2$ and $a$ is a integer.

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## 1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively. A numerical semigroup $S$ is subset of $\mathbb{N}$ that is closed under addition,$0 \in S$, and generates $\mathbb{Z}$ as a group. There exist elements of $S$, say $n_0, n_1, \cdots, n_p$ such that $n_0 < n_1 < \cdots < n_p$ and

$$S = \langle n_0, n_1, \cdots, n_p \rangle = \{\sum_{i=0}^{p} k_i n_i \quad : k_i \in \mathbb{N}\}$$

and

$$g.c.d.(n_0, n_1, \cdots, n_p) = 1 \Leftrightarrow Card(\mathbb{N} \backslash S) < \infty$$

by [1].

The *Frobenius number* of $S$, denoted by $g(S)$, is the largest integer not in $S$. That is, $g(S) = max\{x \in \mathbb{Z} : x \notin S\}$. Thus, a numerical semigroup $S$ can be expressed as $S = \{0, n_0, n_1, \cdots, g(S) + 1 \rightarrow \cdots\}$ where "$\rightarrow$" means that every integer is greater then $g(S) + 1$ belongs to $S$. We say that a numerical semigroup $S$ is *symmetric* if for every $x \in \mathbb{Z} \backslash S$ we have $g(S) - x \in S$ by [1].

The elements of $\mathbb{N} \backslash S$, denoted by $H(S)$, are called *gaps* of $S$. A gap $x$ of a numerical semigroup $S$ is *fundamental* if $\{2x, 3x\} \subset S$. Denote by $FH(S)$ the set of fundamental gaps of $S$ by [2]. Let $a, b \in \mathbb{N}$ and $b \neq 0$. We denote by $a \, mod \, b$ the remainder of division of $a$ by $b$.

A Modular Diophantine inequality is an expression of the form $ax \, mod \, b \leq x$. We say that a numerical semigroup $S$ is *modular* with modulus $b$ and factor $a$ if $S = \{x \in \mathbb{N} : ax \, mod \, b \leq x\}$ by [3]. In this study, we will show that the modular numerical semigroup $S = S(a, a^2)$ will be rewritten as $S = \langle a, a+1 \rangle$, for $a \geq 2$ and $a$ is a integer. In additional, for $S$ modular numerical semigroup, we calculate Frobenius number, the gaps and fundamental gaps of $S$, in a different way.

## 2. Main results

In this section, we will give some results for the theory of modular numerical semigroups generated by two elements.

Lemma 1. *Let $S = S(a, a^2)$ be a modular numerical semigroup. If $x \neq 0, x \in S$, then $x \geq a$ and $a \in \mathbb{N}$.*

*Proof.* We suppose that $x < a$ for $x \in S$. Then there exists $r > 0, r \in \mathbb{N}$ such that $a = x+r$. Thus, we write that $ax = x^2+xr$ and $((x^2+xr) \, mod \, (x^2 + r^2 + 2xr)) = x^2 + xr \leq x$, for $x \in S = S(a, a^2)$. This is a contradiction.

Theorem 2. *If $S = S(a, a^2)$ is a modular numerical semigroup,then $S =< a, a+1 >$ and $S$ is symmetric, for $a \geq 2$ and $a \in \mathbb{N}$ .*

*Proof.* Let $x \in S = S(a, a^2)$. From Lemma 1, $x \geq a$. Then there exist $p, q \in \mathbb{N}$ such that $x = ap + aq + q$. In this case,we get $x \in < a, a+1 >$. Conversely, if $y \in < a, a+1 >$, then we write $y = ak + (a+1)n$ for $k, n \in \mathbb{N}$. Thus,we find that $y \in S(a, a^2)$ since $ay \, mod \, a^2 = an \, mod \, a^2 \leq (ak+n)+an = y$. Furthermore,the modular numerical semigroup $S = S(a, a^2) =< a, a+1 >$ is symmetric since $S$ is generated by two elements [see 5].

Lemma 3. *Let $S = S(a, a^2) =< a, a+1 >$ be a modular numerical semigroup, for $a \geq 2$ and $a \in \mathbb{N}$. Then, $\sharp(H(S)) = \frac{a^2-a}{2}$.*

*Proof.* We obtain that $\sharp(H(S)) = \frac{(a-1)((a+1)-1)}{2} = \frac{a^2-a}{2}$ since $S$ is generated by two elements [ see 4].

Theorem 4. *Let $S = S(a, a^2) =< a, a+1 >$ be a modular numerical semigroup, for $a \geq 2$ and $a \in \mathbb{N}$. Then, $g(S)) = a^2 - a - 1$.*

*Proof.* Since the modular numerical semigroup $S = S(a, a^2) =< a, a+1 >$ is symmetric,we can write that $\sharp(H(S)) = \frac{g(S)+1}{2}$ [ see 4]. Thus, the proof is obtained.

THEOREM 5. *Let $S = S(a, a^2) =< a, a + 1 >$ be a modular numerical semigroup, for $a \geq 4$ and $a \in \mathbb{N}$. If $x \in FH(S)$, then $x > a$.*

*Proof.* If $x = a$, then $x \in S$. This is in contradiction with the definition of Fundamental gaps of $S$. If $x < a$, then we write $a = x + r, r \in \mathbb{N}$. In this case, we obtain that $x^2 + xr > x$. This is a contradiction, since $a \in S, x \notin S$.

COROLLARY 6. *Let $S = S(a, a^2) =< a, a + 1 >$ be a modular numerical semigroup, for $a \geq 2$.and $a \in \mathbb{N}$. If $a$ is odd, then $\sharp(FH(S)) = \frac{(a^2-1)}{4} - \lceil\frac{a+1}{6}\rceil\lceil\frac{a-3}{6}\rceil$. (For a rational number $x$, we define $\lceil x\rceil = min\{z \in \mathbb{Z} : x \leq z\}$).*

*Proof.* If we put $n_2 = a + 1$ in [ 4, Corollary 11(a)], then the proof is obtained.

LEMMA 7. *Let $S_1 = S_1(a, a^2) =< a, a + 1 >$ and $S_2 = S_2(b, b^2) =< b, b + 1 >$ be two modular numerical semigroups for $a, b \in \mathbb{N}$. If $a|b$ then $S_2 \subseteq S_1$.*

*Proof.* This result follows from the fact that $S_1$ and $S_2$ are modular numerical semigroups.

COROLLARY 8. *Let $S_1 = S_1(a, a^2) =< a, a + 1 >$ and $S_2 = S_2(b, b^2) =< b, b + 1 >$ be two modular numerical semigroups for $a, b \in \mathbb{N}$. If $a|b$ then $H(S_1) \subseteq H(S_2)$ and $g(S_1) \leq g(S_2)$.*

*Proof.* This follows immediately from the definitions of gaps and Frobenius number of $S$.

EXAMPLE 9. Let

$$S = S(3, 9) = \{x \in \mathbb{N} : 3x \, mod 9 \leq x\} = \{0, 3, 4, 6, 7, \rightarrow ...\} =< 3, 4 > .$$

By Lemma 3, Theorem 4 and Corollary 6, we obtain that $\sharp(H(S)) = 3$ , $g(S) = 5$ and $\sharp(FH(S)) = 2$. In fact,one can easily compute $H(S) = \{1, 2, 5\}$ and $FH(S) = \{2, 5\}$.

EXAMPLE 10. Let

$$S_1 = S_1(3, 9) = \{x \in \mathbb{N} : 3x \, mod \, 9 \leq x\} = \{0, 3, 4, 6, 7, \rightarrow ...\} =< 3, 4 >$$

and

$$S_2 = S_2(6,36) = \{x \in \mathbb{N} : 6x \, mod \, 36 \leq x\} =$$
$$\{0,6,7,12,13,14,18,19,20,21,24,25,26,27,28,30,31,\rightarrow ...\} =< 6,7 >$$

be two modular numerical semigroups. We obtain that

$$\sharp(H(S_2)) = 15, g(S_2) = 29$$

by Lemma 3 and Theorem 4. In fact, we find sets of

$$H(S_2) = \{1,2,3,4,5,8,9,10,11,15,16,17,22,23,29\}$$

and

$$FH(S_2) = \{9,10,15,16,17,22,23,29\},$$

respectively. We find that $x > 6$,for all $x \in FH(S_2)$. by Theorem 5. Thus, we write that $S_2 \subseteq S_1, H(S_1) \subseteq H(S_2)$ and $g(S_1) \leq g(S_2)$ from Lemma 7, Corollary 8 and Example 9, respectively.

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