SOME REMARKS ABOUT FUZZY INTEGRATION

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Abstract. Symmetrically with the idea of Fuzzy Differentiation, we need a concept and a powerful mathematical tool: the Fuzzy Integration. Previously, we need to analyze the different types of fuzziness acting on functions, their domains, the way to the fuzziness from the independent to dependent variables, the apparition of Fuzzy Bunch of Functions and so on.

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1. Introduction

The first step will be the analysis of different aspects as appear the fuzziness appears into the functions, according to the characteristics of such applications, their domain and range.

The second of such steps will be the study of adequate ways of integration (fuzzy integration).

2. Analyzing the fuzziness of functions

We can find three different kinds of fuzzy functions:

1) *Crisp Function propagating the fuzziness,* from the independent variable to dependent variable.

2) *Crisp Function with fuzzy constraint.*

3) *Fuzzy Function in itself.*

As *first case*, when f is a single fuzzifying function:

Suppose defined $f : U \rightarrow V$, crisp function between universal sets. The fuzzy extension function propagates the fuzziness, or ambiguity, from the independent variable to dependent variable. Such extension function defines the image, $f\left(\tilde{U}\right)$, of fuzzy set $\tilde{U}$.

So, if we suppose the crisp function:

$$f(x) = 5\tilde{x} + 2$$

with domain:

$$A = \{0|0.8,\ 1|0.7,\ 2|0.6,\ 3|0.5\}$$

and range:

$$[2,\ 17]$$

The propagation of fuzziness reaches B, giving a new fuzzy set:

$$B^{*} = \{2|0.8,\ 7|0.7,\ 12|0.6,\ 17|0.5\}$$

The fuzzifying function, $\widetilde{f}$, is a mapping from the same domain to a new range: the family of parts, subsets or power set of the old range:

$$\tilde{f} : U \rightarrow \tilde{P}(V)$$

Such fuzzifying function will be identified with a fuzzy relation, R, on $U \times V$, in this way:

$$\mu_{\widetilde{f(x)}}(y) = \mu_R(x,\ y),\ \forall (x,\ y) \in U \times V$$

Example. Let $A = \{2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8, 9, 12\}$ be two crisp sets. And suppose also the fuzzifying function defined from the elements of A to the power set $\tilde{P}(B)$, established by:

$$\tilde{f}(2) = B_1,\ \tilde{f}(3) = B_2,\ \tilde{f}(4) = B_3$$

with:

$$\tilde{P}(B) = \{B_1,\ B_2,\ B_3\} = \left\{\tilde{f}(2),\ \tilde{f}(3),\ \tilde{f}(4)\right\}$$

being:

$$B_1 = \{2|0.5, 4|1, 6|0.5\}\,,\ B_2 = \{3|0.5, 6|1, 9|0.5\}\,,\ B_3 = \{4|0.5, 8|1, 12|0.5\}$$

So, for instance:

$$\tilde{f} : 2 \in A \to B_1$$
$$\tilde{f} : 3 \in A \to B_2$$
$$\tilde{f} : 4 \in A \to B_3$$

And also it is possible to introduce the $\alpha - cut$ operation on the fuzzifying function:

$$f : 2 \to \left\{ \begin{array}{r} \{2,4,6\}\,, if\ 0 \leq \alpha \leq 0.5 \\ \{4\}\,, if\ 0.5 \leq \alpha \leq 1 \end{array} \right.$$

$$f : 3 \to \left\{ \begin{array}{r} \{3,6,9\}\,, if\ 0 \leq \alpha \leq 0.5 \\ \{6\}\,, if\ 0.5 \leq \alpha \leq 1 \end{array} \right.$$

$$f : 4 \to \left\{ \begin{array}{r} \{4,8,12\}\,, if\ 0 \leq \alpha \leq 0.5 \\ \{8\}\,, if\ 0.5 \leq \alpha \leq 1 \end{array} \right.$$

corresponding to B_1, B_2 and B_3,respectively.

As *second case,* when the function is with fuzzy constraints: Let U and V be universal and crisp sets. And let $A \subseteq U$ and $B \subseteq V$ be fuzzy subsets of such universal sets. We will consider the function:

$$f : U \to V$$

If such function verifies the condition, on the membership degrees of the elements in the domain of f:

$$\mu_A(x) \leq \mu_B(f(x))\,,\ \forall x \in A$$

then we call f a *function with constraints.*

Observe the fuzzy character of both, domain and range of f.

As example: Let $A = \{1|0.3,\ 2|0.6\}$ and $B = \{3|0.5,\ 6|0.7\}$ be two fuzzy sets, and we consider the function: $y = f(x) = 3x$, with $x \in A$ and $y \in B$. In this case, the previous condition holds, because:

if $x = 1$, then $y = 3$, being: $\mu_A(1) = 0.3 \le \mu_B(f(1)) = \mu_B(3) = 0.5$

and

if $x = 2$, then $y = 6$, being: $\mu_A(2) = 0.6 \le \mu_B(f(2)) = \mu_B(6) = 0.7$

In the third case, we depart from a fuzzy set of crisp functions:

$$\{f_i\}_{i=1}^{n}$$

with:

$$f_i : U \to V$$

$$f_i | \mu_{\tilde{f}}(f_i)$$

Then, we define the *Fuzzy Bunch of Functions* as:

$$\tilde{f} \equiv \left\{ f_i \mid \mu_{\tilde{f}}(f_i) \right\}$$

In particular, when $n = 3$, we have:

$$\tilde{f} \equiv \left\{ f_1 \mid \mu_{\tilde{f}}(f_1),\ f_2 \mid \mu_{\tilde{f}}(f_2),\ f_3 \mid \mu_{\tilde{f}}(f_3) \right\}$$

3. Fuzzy Integration

Firstly, we analyze the integration of fuzzifying functions in non-fuzzy intervals, and then, also in fuzzy intervals.

Let $[a,\ b] \subset \mathbb{R}$ be a crisp interval; therefore, non-fuzzy. And let $\tilde{f}(x)$ be the fuzzy value of fuzzifying function, where $x \in [a,\ b]$.

Then, we denote the *Integral* of such fuzzifying function in this interval as:

$$\int_a^b \tilde{f} \ \text{ or } \ \tilde{I}(a,\ b)$$

and defined as:

$$\tilde{I}\left(a,\ b\right) \equiv \left\{ \left[\int_a^b f_\alpha^-\left(x\right) dx + \int_a^b f_\alpha^+\left(x\right) dx \right] |\alpha \right\}_{\alpha\in[0,1]}$$

where f_α^- and f_α^+ are $\alpha - cut$ functions of $\tilde{f}\left(x\right)$.

We can reach the $\alpha - cut$ functions, f_α^- and f_α^+, from the fuzzifying function.

It is possible to integrate both functions, separately:

$$\tilde{I}_\alpha^- = \int_a^b f_\alpha^-\left(x\right) dx,$$

$$\tilde{I}_\alpha^+ = \int_a^b f_\alpha^+\left(x\right) dx.$$

So, we conclude that the possibility of $\tilde{I}_\alpha^-$ to be a member of $\tilde{I}\left(a,\ b\right)$ - the total integration - is equal to α. And analogously: the possibility of $\tilde{I}_\alpha^+$ to be a member of $\tilde{I}\left(a,\ b\right)$ is equal to α.

As example, let:

$$\tilde{f} = \{f_1|0.2,\ f_2|0.3,\ f_3|0.3,\ f_4|0.5\}$$

in the universal set:

$$U = [3,\ 5]$$

a crisp interval, with:

$$f_1\left(x\right) = 2x + 1,$$
$$f_2\left(x\right) = 3x^2 + 2,$$
$$f_3\left(x\right) = 4x^3 + 3,$$
$$f_4\left(x\right) = 5x^4 + 4.$$

We will obtain the integration of such fuzzy function.

With $f = f_1$, the integral at $\alpha = 0.2$ will be:

$$I_\alpha(3,5) = \int_3^5 (2x+1)\, dx = \left[x^2 + x\right]_3^5 = 18.$$

Therefore:

$$\tilde{I}_{0.2}(3,5) = \{18|0.2\}.$$

With $f = f_2$, the integral at $\alpha = 0.3$ will be:

$$I_\alpha(3,5) = \int_3^5 \left(3x^2+2\right) dx = \left[x^3 + 2x\right]_3^5 = 102.$$

Therefore:

$$\tilde{I}_{0.3}(3,5) = \{102|0.3\}.$$

With $f = f_3$, the integral at $\alpha = 0.4$ will be:

$$I_\alpha(3,5) = \int_3^5 \left(4x^3+3\right) dx = \left[x^4 + 3x\right]_3^5 = 550.$$

Therefore:

$$\tilde{I}_{0.4}(3,5) = \{550|0.4\}.$$

With $f = f_4$, the integral at $\alpha = 0.5$ will be:

$$I_\alpha(3,5) = \int_3^5 \left(5x^4+4\right) dx = \left[x^5 + 4x\right]_3^5 = 2890.$$

Therefore:

$$\tilde{I}_{0.5}(3,5) = \{2890|0.5\}.$$

So, the final integration will be:

$$\tilde{I}(3,\ 5) = \{18|0.2, 102|0.3, 550|0.4, 2890|0.5\}.$$

The other situation will be when integrating crisp function in fuzzy interval, $[A,\ B]$, with boundaries determined by A and B, fuzzy sets.

We denote such integration $I(A, B)$, which is defined through the membership function:

$$\mu_{I(A,\ B)}(z) = \max_{z=\int_x^y f(u)du} [\min\{\mu_A(x), \mu_B(y)\}]$$

As illustrative *example*, we shows the integration of $f(x) = 2$, defined on the fuzzy interval $[A, B]$, with $A = \{4|0.8, 5|1, 6|0.4\}$ and $B = \{6|0.7, 7|1, 8|0.2\}$. Then, we have:

$[a, b]$	$z = \int_a^b 2dx = 2(b-a)$	$\min\{\mu_A(a), \mu_B(b)\}$;
$[4, 6]$	4	$\min\{\mu_A(4), \mu_B(6)\} = 0.7$;
$[4, 7]$	6	$\min\{\mu_A(4), \mu_B(7)\} = 0.8$;
$[4, 8]$	8	$\min\{\mu_A(4), \mu_B(8)\} = 0.2$
$[5, 6]$	2	$\min\{\mu_A(5), \mu_B(6)\} = 0.7$;
$[5, 7]$	4	$\min\{\mu_A(5), \mu_B(7)\} = 1$;
$[5, 8]$	6	$\min\{\mu_A(5), \mu_B(8)\} = 0.2$;
$[6, 6]$	0	$\min\{\mu_A(6), \mu_B(6)\} = 0.4$;
$[6, 7]$	2	$\min\{\mu_A(6), \mu_B(7)\} = 0.4$;
$[6, 8]$	4	$\min\{\mu_A(6), \mu_B(8)\} = 0.2$.

So, the integration values are:

$$\tilde{I}(A,\ B) = \{0|0.4,\ 2|0.7,\ 4|1,\ 6|0.8,\ 8|0.2\}.$$

Observe that integrating in $[5, 6]$ and $[6, 7]$,we obtain the integration value 2, with possibilities equal to 0.4 and 0.7, respectively. In such case, we take as possibility to be 2 the integration value by the maximum between both values:

$$\text{Max}\{0.4, 0.7\} = 0.7 = \text{membership degree of 2 as element of } \tilde{I}(A,\ B).$$

In general, we must obviously distinguish between cases, according to the presence of a *crisp domain* or a *fuzzy domain, D,* for the function f.

Let $\widetilde{f}$ be a fuzzifying function on $[A,\ B]$, fuzzy interval, where A and B will be fuzzy sets. Then, we have:

$$\mu_{I(\widetilde{A,}\ B)}(Z) = \max[\min\{\mu_A(X), \mu_B(Y)\}].$$

That is:

$$Z = I_{\widetilde{f}}(X,Y) = \int_X^Y f_{\alpha^+}(u)\ du + \int_X^Y f_{\alpha^-}(u)\,du$$

where f_{α^+} and f_{α^-} are α-cut functions of the fuzzifying function $\widetilde{f}$, or Fuzzy Bunch: $\{f_i\}_{i \in I}$.

Therefore:

$$\exists i_1 \in I : f_{\alpha^+} = f_{i_1} \in \widetilde{f} \qquad \text{and} \qquad \exists i_2 \in I : f_{\alpha^-} = f_{i_2} \in \widetilde{f}.$$

So:

$$\widetilde{I}(A,B) = \left\{ z \mid \mu_{\widetilde{I}(A,B)}(z) \right\}_{z \in U} = \left\{ I_{\widetilde{f}}^{+} + I_{\widetilde{f}}^{-} \mid \alpha \in [0,1] \right\}.$$

In the aforementioned case:

$$\widetilde{I}(A,B) = \{ z \mid \max[\min\{\mu_A(X), \mu_B(Y)\}]\}$$

with: $X \subseteq A$, $Y \subseteq B$, being:

$$Z = \int f_{i_1} + \int f_{i_2}.$$

For instance, let:

$$A = \{1|0.1,\ 2|0.2,\ 3|0.3\} \quad \text{and} \quad B = \{4|0.4,\ 5|0.5,\ 6|0.6\}$$

be fuzzy sets. Also, let:

$$\widetilde{f} = \{f_1|0.2,\ f_2|0.4\}$$

be a fuzzy bunch of functions.

Then, we have the fuzzy interval $[A,\ B]$.

Taking:

$$f_{\alpha^+} = f_1$$
$$f_{\alpha^-} = f_2$$

we obtain:

$$z = \int f_1 + \int f_2 = \int f_{\alpha^+} + \int f_{\alpha^-}.$$

We obtain the subsequent *Fuzzy Integration Table:*

$[a,\ b]$	$\int_A^B \widetilde{f}$	$\min\{\mu_A(X),\ \mu_B(Y)\};$
$[1,\ 4]$	17	$\min\{\mu_A(1),\ \mu_B(4)\} = 0.1;$
$[1,\ 5]$	$\frac{43}{2}$	$\min\{\mu_A(1),\ \mu_B(5)\} = 0.1;$
$[1,\ 6]$	27	$\min\{\mu_A(1),\ \mu_B(6)\} = 0.1;$
$[2,\ 4]$	7	$\min\{\mu_A(2),\ \mu_B(4)\} = 0.2;$
$[2,\ 5]$	$\frac{23}{2}$	$\min\{\mu_A(2),\ \mu_B(5)\} = 0.2;$
$[2,\ 6]$	17	$\min\{\mu_A(2),\ \mu_B(6)\} = 0.2;$
$[3,\ 4]$	$\frac{7}{2}$	$\min\{\mu_A(3),\ \mu_B(4)\} = 0.3;$
$[3,\ 5]$	8	$\min\{\mu_A(3),\ \mu_B(5)\} = 0.3;$
$[3,\ 6]$	$\frac{27}{2}$	$\min\{\mu_A(3),\ \mu_B(6)\} = 0.3.$

As consequence, in a first step:

$$\widetilde{I}(A,B) = \left\{17|0.1, \tfrac{43}{2}|0.1, 27|0.1, 7|0.2, \tfrac{23}{2}|0.2, 17|0.2, \tfrac{7}{2}|0.3, 8|0.3, \tfrac{27}{2}|0.3\right\}.$$

But we will consider the maximum membership degree, when it the same element appears, as in the case of $x = 17$, where:

$$17|\max\{0.1, 0.2\} = 17|0.2.$$

Therefore, by substitution and reordering:

$$\widetilde{I}(A,B) = \{3.5|0.3,\ 7|0.2,\ 8|0.3,\ 11.5|0.2,\ 13.5|0.3,\ \ 17|0.2,\ 21.5|0.1,\ 27|0.1\}.$$

4. Conclusion

I hope to contribute with these remarks to the study of Fuzzy Integration and therefore, of Fuzzy Measure.

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