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INCLUSION RELATIONSHIPS PROPERTIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH HURWITZ-LERECH ZETA FUNCTION

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Abstract. Let Σ denote the class of analytic functions in the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. In this paper, we introduce several new subclasses of meromorphic functions defined by means of the linear operator ${}^{s}_{d}(a, c; z)$. Inclusion properties of these classes and some applications involving integral operator are also considered.

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*sc*1. Introduction

Let Σ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$.

Function $f \in \Sigma$ is said to be in the class $\Sigma S^*(\alpha)$ of meromorphic starlike functions of order α in U^* if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in U^*; 0 \le \alpha < 1).$$

Also a function $f \in \Sigma$ is said to be in the class $\Sigma C(\alpha)$ of meromorphic convex of order α in U^* if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in U^*; 0 \le \alpha < 1).$$

It is easy to observe that

$$f(z) \in \Sigma C(\alpha) \Leftrightarrow -zf^{'}(z) \in \Sigma S^{*}(\alpha)\,. \tag{1.2}$$

For a function $f \in \Sigma$, we say that $f \in \Sigma K(\beta,\alpha)$ if there exists a function $g \in \Sigma S^{*}(\alpha)$ such that

$$Re\left(\frac{zf^{'}(z)}{g(z)}\right) < -\beta \quad (z \in U^{*}; 0 \leq \alpha, \beta < 1)\,.$$

Functions in the class $\Sigma K(\beta,\alpha)$ are called meromorphic close-to-convex functions of order β and type α (see [2], [6], [9], [14], [15]). We also say that a function $f \in \Sigma$ is in the class $\Sigma K^{*}(\beta,\alpha)$ of meromorphic quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C(\alpha)$ such that

$$Re\left(\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right) < -\beta \quad (z \in U^{*}; 0 \leq \alpha, \beta < 1)\,.$$

Also, it is easy to observe that

$$f(z) \in \Sigma K^{*}(\beta,\alpha) \Leftrightarrow -zf^{'}(z) \in \Sigma K(\beta,\alpha)\,. \tag{1.3}$$

For two functions $f_j(z) \in \Sigma$ $(j = 1,2)$, given by

$$f_j(z) = \frac{1}{z} +_{n=0}^{\infty} a_{n,j} z^{n} \quad (j = 1,2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} +_{n=0}^{\infty} a_{n,1} a_{n,2} z^{n} = (f_2 * f_1)(z).$$

The general Hurwitz-Lerech Zeta function $\Phi(z,s,b)$ defined by (see [16])

$$\Phi(z,s,d) = \sum_{n=0}^{\infty} \frac{z^{n}}{(n+d)^{s}}$$

$$(d \in \mathbb{C}\backslash\mathbb{Z}_0^{-} = \{0,-1,-2,....\}; s \in \mathbb{C} \text{ when } |z| < 1; Res > 1 \text{ when } |z| = 1). \tag{1.4}$$

Several interesting properties and characteristics of Hurwitz-Lerech Zeta function $\Phi(z,s,d)$ can found in the investigations by several authors (see [4], [5], [10], [11]).

Now, we define the function $H_d^{s}(z)$ $(d \in \mathbb{C}\backslash\mathbb{Z}_0^{-}; s \in \mathbb{C})$ by

$$H_d^{s}(z) = \frac{d^{s}}{z}\Phi(z,s,d) \quad (z \in U^{*}). \tag{1.5}$$

We also denote by

$${}^s_d f(z) : \Sigma \to \Sigma,$$

the linear operator defined by

$${}^s_d f(z) = H^s_d(z) * f(z) \quad (d \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; z \in U^*).$$

We note that

$${}^s_d f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left(\frac{d}{n+d+1} \right)^s a_n z^n. \tag{1.6}$$

Also we note that

(i) ${}^\alpha_\beta f(z) = P^\alpha_\beta f(z)$ $(\alpha, \beta > 0)$ (see Lashin [8]);

(ii) ${}^\alpha_1 f(z) = P^\alpha f(z)$ $(\alpha > 0)$ (see Aqlan et al. [1], with $p = 1$);

(iii) ${}^1_\mu(f(z) = J_\mu f(z)$ $(\mu > 0)$ (see [12, p.11 and 389]).

Finally, for $f(z) \in \Sigma, z, t_i \in U^*(i = 1, 2, ..., n),$ $n \in \mathbb{N}$ and $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, we have

${}^0_1 f(z) = f(z)$ and ${}^0_d f(z) = f(z)$

$${}^1_1 f(z) = \frac{1}{z^2}\int_0^z t_1 f(t_1) dt_1 \quad (f \in \Sigma; z \in U^*)$$

$${}^2_1 f(z) = \frac{1}{z^2}\int_0^z \frac{1}{t_1}\int_0^{t_1} t_2 f(t_2) dt_2 dt_1 \quad (f \in \Sigma; z \in U^*)$$

....

$${}^n_1 f(z) = \frac{1}{z^2}\int_0^z \frac{1}{t_1}\int_0^{t_1} \frac{1}{t_2}\int_0^{t_2} \cdots \frac{1}{t_{n-1}}\int_0^{t_{n-1}} t_n f(t_n) dt_n \ dt_{n-1} ... dt_2 dt_1 \ (f \in \Sigma; z \in U^*)$$

$${}^1_d f(z) = \frac{d}{z^{d+1}}\int_0^z t^d f(t) dt \quad (f \in \Sigma; z \in U^*)$$

$${}^2_d f(z) = \frac{d^2}{z^{d+1}}\int_0^z \frac{1}{t_1}\int_0^{t_1} t_2^d f(t_2) dt_2 dt_1 \quad (f \in \Sigma; z \in U^*)$$

$${}^n_d f(z) = \frac{d^n}{z^{d+1}}\int_0^z \frac{1}{t_1}\int_0^{t_1} \frac{1}{t_2}\int_0^{t_2} \cdots \frac{1}{t_{n-1}}\int_0^{t_{n-1}} t_n^d f(t_n) dt_n \ dt_{n-1} ... dt_2 dt_1 \ (f \in \Sigma; z \in U^*).$$

Also, we can show that

$${}^{s+1}_d f(z) = \frac{d}{z^{d+1}}\int_0^z t^{d\,s}_d f(t) dt \quad (f \in \Sigma; z \in U^*).$$

Let us define the function

$$\Psi(a, c; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} z^n \quad (a \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}; c \in \mathbb{C}\backslash\mathbb{Z}_0^-; z \in U^*), \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1).........(\lambda + n - 1) & (n \in N). \end{cases}$$

We not that

$$\Psi(a, c; z) = \frac{1}{z} \, {}_2F_1(a, 1; c; z),$$

where

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \quad (a,b,c \in \mathbb{C} \text{ and } c \notin \mathbb{Z}_0^-; z \in U), $$

is the (Gaussian) hypergeometric function.

Set

$$ {}^{n}_{d}(z) *^{s}_{d} (z) = \frac{1}{z\,(1-z)}, $$

we, obtain

$$ {}^{s}_{d}(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left(\frac{n+d+1}{d} \right)^{s} z^n. $$

Now, we define the operator ${}^{s}_{d}(a,c;z)$ as follows:

$$ {}^{s}_{d}(z) *^{s}_{d} (a,c;z) = \Psi(a,c;z) \qquad (z \in U^*). \tag{1.8} $$

The linear operator ${}^{s}_{d}(a,c;z) : \Sigma \to \Sigma$, is defined here by:

$$ {}^{s}_{d}(a,c;z)f(z) ={}^{s}_{d} (a,c;z) * f(z) \ (s \in \mathbb{C}; a \in \mathbb{C}^*; c,d \in \mathbb{C}\backslash\mathbb{Z}_0^-). \tag{1.9} $$

It is easily verified from the definition of the operator ${}^{s}_{d}(a,c;z)$, that

$$ z({}^{s+1}_{d}(a,c;z)f(z))' = d{}^{s}_{d}(a,c;z)f(z) - (d+1){}^{s+1}_{d}(a,c;z)f(z) \tag{1.10} $$

and

$$ z({}^{s}_{d}(a,c;z)f(z))' = a{}^{s}_{d}(a+1,c;z)f(z) - (a+1){}^{s}_{d}(a,c;z)f(z) \ (a \in \mathbb{C}\backslash\{-1\}). \tag{1.11} $$

We note that
${}^{s}_{d}(\mu,1;z)f(z) = I^{s}_{d,\mu}f(z)$ $(d,\mu \in \mathbb{R}^+, s \in \mathbb{N}_0)$ (see Cho et al. [3]).

Next, by using the operator ${}^{s}_{d}(a,c;z)$ defined by (1.9), we introduce the following subclasses of meromorphic functions:

$$ \Sigma S^{s,a}_{c,d}(\eta) = \left\{ f : f \in \Sigma \text{ and } {}^{s}_{d}(a,c;z)f(z) \in \Sigma S^*(\alpha)\,, 0 \le \alpha < 1 \right\}. $$

$$ \Sigma C^{s,a}_{c,d}(\alpha) = \left\{ f : f \in \Sigma \text{ and } {}^{s}_{d}(a,c;z)f(z) \in \Sigma C(\alpha), 0 \le \alpha < 1 \right\}. $$

We note that

$$ f(z) \in \Sigma C^{s,a}_{c,d}(\alpha) \Leftrightarrow -zf'(z) \in \Sigma S^{s,a}_{c,d}(\eta). \tag{1.12} $$

$$ \Sigma K^{s,a}_{c,d}(\beta,\alpha) = \left\{ f : f \in \Sigma \text{ and } {}^{s}_{d}(a,c;z)f(z) \in \Sigma K(\beta,\alpha), 0 \le \alpha,\beta < 1 \right\}. $$

$$ \Sigma K^{*s,a}_{c,d}(\beta,\alpha) = \left\{ f : f \in \Sigma \text{ and } {}^{s}_{d}(a,c;z)f(z) \in \Sigma K^*(\beta,\alpha)\,, 0 \le \alpha,\beta < 1 \right\}. $$

Also, we note that

$$f(z) \in \Sigma K^{*s,a}_{c,d}(\beta,\alpha) \Leftrightarrow -zf^{'}(z) \in \Sigma K^{s,a}_{c,d}(\beta,\alpha). \quad (1.13)$$

In order to establish our main results, we need the following lemma due to Miller and Mocanu [13].

Lemma 1 [13]**.** *Let $\theta(u,v)$ be a complex-valued function such that*

$$\theta : D \to \mathbb{C},\, D \subset \mathbb{C}\times\mathbb{C} \quad (\mathbb{C} \text{ is the complex plane})$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions :

(i) $\theta(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\{\theta(1,0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that

$$v_1 \le -\frac{1}{2}(1+u_2^2) \quad , \quad Re\{\theta(iu_2,v_1)\} \le 0\,.$$

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + ... \quad (1.14)$$

be analytic in U such that $(q(z), zq^{'}(z)) \in D\,(z \in U)$. If

$$Re\left\{\theta(q(z), zq^{'}(z))\right\} > 0 \quad (z \in U)\,,$$

then

$$Re\{q(z)\} > 0 \quad (z \in U)\,.$$

The main object of this paper is to investigate several inclusion properties of the classes mentioned above. Some applications involving integral operator are also considered.

2. Inclusion properties involving the operator $^{s}_{d}$(a,c;z)

Unless otherwise mentioned we shall assume throughout the paper that $s \in \mathbb{C}, a \in \mathbb{C}^*$ and $c,d \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 1. Let $a_1 = Re(a) > \alpha - 1, d_1 = Re(d) > \alpha - 1$ and $0 \le \alpha < 1$, *then*

$$\Sigma S^{s,a+1}_{c,d}(\alpha) \subset \Sigma S^{s,a}_{c,d}(\alpha) \subset \Sigma S^{s+1,a}_{c,d}(\alpha).$$

We begin by showing the first inclusion relationship:

$$\Sigma S^{s,a+1}_{c,d}(\alpha) \subset \Sigma S^{s,a}_{c,d}(\alpha), \quad (2.1)$$

which is asserted by Theorem 1. Let $f \in \Sigma S_{c,d}^{s,a+1}(\alpha)$ and set

$$q(z) = \frac{1}{1-\alpha}\left(-\frac{z(_{d}^{s}(a,c;z)f(z))'}{_{d}^{s}(a,c;z)f(z)} - \alpha\right), \tag{2.2}$$

where $q(z)$ is given by (1.14). Then, by applying equations (1.11) in (2.2), we obtain

$$a\frac{_{d}^{s}(a+1,c;z)f(z)}{_{d}^{s}(a,c;z)f(z)} = -(1-\alpha)q(z) + (a+1-\alpha). \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to z and multiplying the resulting equation by z, we have

$$\frac{z(_{d}^{s}(a+1,c;z)f(z))'}{_{d}^{s}(a+1,c;z)f(z)} = -\alpha - (1-\alpha)q(z) + \frac{(1-\alpha)zq'(z)}{(1-\alpha)q(z)+\alpha-(a+1)} \quad (z \in U).$$

Let

$$\theta(u,v) = (1-\alpha)u - \frac{(1-\alpha)v}{(1-\alpha)u+\alpha-(a+1)} \tag{2.4}$$

with $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$. Then

(i) $\theta(u,v)$ is continuous in $D = \left(\mathbb{C}\backslash\left\{\frac{a+1-\alpha}{1-\alpha}\right\}\right) \times \mathbb{C}$;

(ii) $(1,0) \in D$ with $\{\theta(1,0)\} = 1-\alpha > 0$.

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le -\frac{1}{2}\left(1+u_2^2\right)$ we have

$$\begin{aligned} Re\{\theta(iu_2,v_1)\} &= Re\left\{\frac{-(1-\alpha)v_1}{(1-\alpha)iu_2+\alpha-(a+1)}\right\} \\ &= \frac{(1-\alpha)(a_1+1-\alpha)v_1}{((1-\alpha)u_2+a_2)^2+(\alpha-a_1-1)^2} \\ &\le -\frac{(1-\alpha)(a_1+1-\alpha)(1+u_2^2)}{2\left([(1-\alpha)u_2+a_2]^2+(a_1-1-\alpha)^2\right)} \\ &< 0, \end{aligned}$$

which shows that $\theta(u,v)$ satisfies the hypotheses of Lemma 1. Consequently, we easily obtain the inclusion relationship (2.1).

(ii) By using arguments similar to those detailed above, together with (1.10) and $\theta(u,v)$ is continuous in $D = \left(\mathbb{C}\backslash\left\{\frac{(d+1)-\alpha}{1-\alpha}\right\}\right) \times \mathbb{C}$, we can also prove the right part of Theorem 1, that is, that

$$S_{c,d}^{s,a}(\alpha) \subset S_{c,d}^{s+1,a}(\alpha). \tag{2.5}$$

Combining the inclusion relationships (2.1) and (2.5), we complete the proof of Theorem 1.
Theorem 2. Let $a_1 = Re(a) > \alpha - 1, d_1 = Re(d) > \alpha - 1$ and $0 \leq \alpha < 1$, *then*

$$\Sigma C_{c,d}^{s,a+1}(\alpha) \subset \Sigma C_{c,d}^{s,a}(\alpha) \subset \Sigma C_{c,d}^{s+1,a}(\alpha).$$

Let $f \in \Sigma K_{c,d}^{s,a+1}(\alpha)$. Then, we have

$$ {}_{d}^{s}(a+1,c;z)f(z) \in \Sigma C(\alpha) \ .$$

Furthermore, in view of the relationship (1.2), we find that

$$-z({}_{d}^{s}(a+1,c;z)f(z))^{'} \in \Sigma S^{*}(\alpha)\,,$$

that is, that

$$ {}_{d}^{s}(a+1,c;z)\left(-zf^{'}(z)\right) \in \Sigma S^{*}(\alpha)\,.$$

Thus, by Theorem 1, we have

$$-zf^{'}(z) \in \Sigma S_{c,d}^{s,a+1}(\alpha) \subset \Sigma S_{c,d}^{s,a}(\alpha)\ ,$$

which implies that

$$\Sigma C_{c,d}^{s,a+1}(\alpha) \subset \Sigma C_{c,d}^{s,a}(\alpha)\,.$$

The right part of Theorem 2 can be proved by using similar arguments. The proof of Theorem 2 is thus completed.
Theorem 3. Let $a_1 = Re(a) > \alpha - 1, d_1 = Re(d) > \alpha - 1$ and $0 \leq \alpha, \beta < 1$, *then*

$$\Sigma K_{c,d}^{s,a+1}(\alpha,\beta) \subset \Sigma K_{c,d}^{s,a}(\alpha,\beta) \subset \Sigma K_{c,d}^{s+1,a}(\alpha,\beta;).$$

Let us begin by proving that

$$\Sigma K_{c,d}^{s,a+1}(\alpha,\beta) \subset \Sigma K_{c,d}^{s,a}(\alpha,\beta)\,. \tag{2.6}$$

Let $f(z) \in \Sigma K_{c,d}^{s,a+1}(\alpha,\beta)$. Then there exists a function $\Psi(z) \in \Sigma S^{*}(\alpha)$ such that

$$Re\left(\frac{z({}_{d}^{s}(a+1,c;z)f(z))^{'}}{\Psi(z)}\right) < -\beta \quad (z \in U)\,.$$

We put

$$ {}_{d}^{s}(a+1,c;z)g(z) = \Psi(z)\ ,$$

so that we have

$$g(z) \in \Sigma S_{c,d}^{s,a+1}(\alpha) \text{ and } Re\left(\frac{z({}_{d}^{s}(a+1,c;z)f(z))'}{{}_{d}^{s}(a+1,c;z)g(z)}\right) < -\beta \quad (z \in U)\,.$$

We next put

$$\frac{z({}_{d}^{s}(a,c;z)f(z))'}{{}_{d}^{s}(a,c;z)g(z)} = -\beta - (1-\beta)q(z)\,, \tag{2.7}$$

where $q(z)$ is given by (1.14). Thus, by using the identity (1.11), we obtain

$$\begin{aligned}\frac{z({}_{d}^{s}(a+1,c;z)f(z))'}{{}_{d}^{s}(a+1,c;z))g(z)} &= \frac{({}_{d}^{s}(a+1,c;z)(zf'(z))}{{}_{d}^{s}(a+1,c;z)g(z)} \\ &= \frac{z\left[{}_{d}^{s}(a,c;z)(zf'(z))\right]' + (a+1){}_{d}^{s}(a,c;z)(zf'(z))}{z({}_{d}^{s}(a,c;z)g(z))' + (a+1){}_{d}^{s}(a,c;z)g(z)} \\ &= \frac{\dfrac{z\left[{}_{d}^{s}(a,c;z)(zf'(z))\right]'}{{}_{d}^{s}(a,c;z)g(z)} + (a+1)\dfrac{{}_{d}^{s}(a,c;z)(zf'(z))}{{}_{d}^{s}(a,c;z)g(z)}}{\dfrac{z({}_{d}^{s}(a,c;z)g(z))'}{{}_{d}^{s}(a,c;z)g(z)} + (a+1)}\,.\end{aligned}$$

Since $g(z) \in \Sigma S_{c,d}^{s,a+1}(\alpha)$, by Theorem 1, we can put

$$\frac{z({}_{d}^{s}(a,c;z)(g(z))'}{{}_{d}^{s}(a,c;z)g(z)} = -\alpha - (p-\alpha)G(z)\,,$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \text{ and } Re(G(z)) = g_1(x,y) > 0 \quad (z \in U)\,.$$

Then

$$\frac{z({}_{d}^{s}(a+1,c;z)f(z))'}{{}_{d}^{s}(a+1,c;z)g(z)} = \frac{\dfrac{z\left[{}_{d}^{s}(a,c;z)(zf'(z))\right]'}{{}_{d}^{s}(a,c;z)g(z)} - (a+1)\left[\beta + (1-\beta)q(z)\right]}{-\alpha - (1-\alpha)G(z) + (a+1)}\,. \tag{2.8}$$

We thus find from (2.7) that

$$z({}_{d}^{s}(a,c;z)f(z))' = -{}_{d}^{s}(a,c;z)g(z)\left[\beta + (1-\beta)q(z)\right]\,. \tag{2.9}$$

Differentiating both sides of (2.9) with respect to z, we obtain

$$\frac{z\left[{}_d^s(a,c;z)zf'(z))\right]'}{{}_d^s(a,c;z)g(z)} = -(1-\beta)zq'(z) + \left[\alpha + (1-\alpha)G(z)\right]\left[\beta + (1-\beta)q(z)\right]. \quad (2.10)$$

By substituting (2.10) into (2.8), we have

$$\frac{z({}_d^s(a+1,c;z)f(z))'}{{}_d^s(a+1,c;z)g(z)} + \beta = -\left\{(1-\beta)q(z) - \frac{(p-\beta)zq'(z)}{(1-\alpha)G(z)+\alpha-(a+1)}\right\}.$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = (1-\beta)u - \frac{(1-\beta)v}{(1-\alpha)G(z)+\alpha-(a+1)}, \quad (2.11)$$

where $(u,v) \in D = (\mathbb{C}\backslash D^*) \times \mathbb{C}$ and

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } Re(G(z)) \geq \frac{Re(a)}{(1-\alpha)} + 1\right\}.$$

Then it follows from (2.11) that

(i) $\Phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\{\Phi(1,0)\} = 1-\beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$Re\{\Phi(iu_2,v_1)\} = Re\left\{-\frac{(1-\beta)v_1}{(1-\alpha)G(z)+\alpha-a-1}\right\}$$

$$\begin{aligned} &= \frac{(1-\beta)v_1\left[a_1+1-\alpha-(1-\alpha)g_1(x,y)\right]}{\left[(1-\alpha)g_1(x,y)+\alpha-a_1-1\right]^2 + \left[(1-\alpha)g_2(x,y)-a_2\right]^2} \\ &\leq -\frac{(1-\beta)(1+u_2^2)\left[a_1+1-\alpha-(1-\alpha)g_1(x,y)\right]}{2\left[(1-\alpha)g_1(x,y)+\alpha-a_1-1\right]^2 + 2\left[(1-\alpha)g_2(x,y)-a_2\right]^2} \\ &< 0, \end{aligned}$$

which shows that $\Phi(u,v)$ satisfies the hypotheses of Lemma 1. Thus, in light of (2.7), we easily deduce the inclusion relationship (2.6).

The remainder of our proof of Theorem 3 would make use of the identity (1.10) in an analogous manner and assume that

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } Re(G(z)) \geq \frac{Re(d)}{(1-\alpha)} + 1\right\}.$$

We, therefore, choose to omit the details involved.
Theorem 4. Let $a_1 = Re(a) > \alpha - 1, d_1 = Re(d) > \alpha - 1$ and $0 \leq \alpha, \beta < 1$, *then*

$$\Sigma K^{*s,a+1}_{c,d}(\beta, \alpha) \subset \Sigma K^{*s,a}_{c,d}(\beta, \alpha) \subset \Sigma K^{*s+1,a}_{c,d}(\beta, \alpha) .$$

Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (1.2), we can also prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (1.3).

3. Inclusion properties involving the integral operator J_μ

In this section, we consider the integral operator J_μ (see, (iii) in the introduction) defined by

$$J_\mu(f)(z) = \frac{\mu}{z^{\mu+1}} \int_0^z t^\mu f(t)dt \quad (\mu > 0; f \in \Sigma) , \tag{3.1}$$

in order to obtain the integral-preserving properties involving the integral operator J_μ.

From the definition (3.1), it is easily verified that

$$z \left({}^{s}_{d}(a, c; z) J_\mu(f)(z)\right)' = \mu {}^{s}_{d}(a, c; z) f(z) - (\mu + 1) {}^{s}_{d}(a, c; z) J_\mu(f)(z) . \tag{3.2}$$

We need the following lemma which is popularly known as Jack's lemma .
Lemma 2 ([7]). *Let $w(z)$ be a non-constant function analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = \zeta w(z_0) ,$$

where $\zeta \geq 1$ is a real number.
Theorem 5. Let $\mu > 0$ and $0 \leq \alpha < 1$. *If* $f(z) \in \Sigma S^{s,a}_{c,d}(\alpha)$, *then*

$$J_\mu(f)(z) \in \Sigma S^{s,a}_{c,d}(\alpha) .$$

Suppose that $f(z) \in \Sigma S^{s,a}_{c,d}(\alpha)$ and let

$$\frac{z \left({}^{s}_{d}(a, c; z) J_\mu(f)(z)\right)'}{{}^{s}_{d}(a, c; z) J_\mu(f)(z)} = -\frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)} , \tag{3.3}$$

where $w(0) = 0$. Then, by using (3.2) and (3.3), we have

$$\frac{{}^{s}_{d}(a, c; z) f(z)}{{}^{s}_{d}(a, c; z) J_\mu(f)(z)} = \frac{\mu - (\mu + 2 - 2\alpha)w(z)}{\mu(1 - w(z))} . \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we obtain

$$\begin{aligned} \frac{z\left({}_{d}^{s}(a,c;z)f(z)\right)'}{{}_{d}^{s}(a,c;z)f(z)} &= -\frac{1+(1-2\alpha)w(z)}{1-w(z)}+\frac{zw'(z)}{1-w(z)} \\ &\quad -\frac{(\mu+2-2\alpha)zw'(z)}{\mu-(\mu+2-2\alpha)w(z)}\,,3.5 \end{aligned} \tag{1}$$

so that

$$\begin{aligned} \frac{z\left({}_{d}^{s}(a,c;z)f(z)\right)'}{{}_{d}^{s}(a,c;z)f(z)}+\alpha &= \frac{(\alpha-1)(1+w(z))}{1-w(z)}+\frac{zw'(z)}{1-w(z)} \\ &\quad -\frac{(\mu+2-2\alpha)zw'(z)}{\mu-(\mu+2-2\alpha)w(z)}.3.6 \end{aligned} \tag{2}$$

Now, assuming that $\max\limits_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1\,(z_0\in U)$ and applying Jack's lemma, we have

$$z_0w'(z_0)=\zeta w(z_0)\quad(\zeta\ge 1)\,. \tag{3.7}$$

If we set $w(z_0)=e^{i\theta}\,(\theta\in R)$ in (3.6) and observe that

$$Re\left\{\frac{(\alpha-1)(1+w(z_0))}{1-w(z_0)}\right\}=0\,,$$

then we obtain

$$\begin{aligned} Re\left\{\frac{z_0\left({}_{d}^{s}(a,c;z)f(z_0)\right)'}{{}_{d}^{s}(a,c;z)f(z_0)}+\alpha\right\} &= Re\left\{\frac{z_0w'(z_0)}{1-w(z_0)}-\frac{(\mu+2-2\alpha)z_0w'(z_0)}{\mu-(\mu+2-2\alpha)w(z_0)}\right\} \\ &= Re\left\{-\frac{2(1-\alpha)\zeta e^{i\theta}}{(1-e^{i\theta})\left[\mu-(\mu+2-2\alpha)e^{i\theta}\right]}\right\} \\ &= \frac{2\zeta(1-\alpha)(\mu+1-\alpha)}{\mu^2-2\mu(\mu+2-2\alpha)\cos\theta+(\mu+2-2\alpha)^2} \\ &\ge 0\,, \end{aligned}$$

which obviously contradicts the hypothesis $f(z)\in\Sigma S_{c,d}^{s,a}(\alpha)$. Consequently, we can deduce that $|w(z)|<1\,(z\in U)$, which, in view of (3.3), proves the integral-preserving property asserted by Theorem 5.

Theorem 6. Let $\mu>0$ and $0\le\alpha<1$. *If* $f(z)\in\Sigma C_{c,d}^{s,a}(\alpha)$, *then*

$$J_\mu(f)(z)\in\Sigma C_{c,d}^{s,a}(\alpha)\,.$$

By applying Theorem 5, it follows that

$$\begin{aligned} f(z) &\in \Sigma C^{s,a}_{c,d}(\alpha) \Leftrightarrow -zf^{'}(z) \in \Sigma S^{s,a}_{c,d}(\alpha) \\ &\Rightarrow J_\mu\left(-zf^{'}(z)\right) \in \Sigma S^{s,a}_{c,d}(\alpha) \\ &\Leftrightarrow -z\left(J_\mu f(z)\right)^{'} \in \Sigma S^{s,a}_{c,d}(\alpha) \\ &\Rightarrow J_\mu(f)(z) \in \Sigma C^{s,a}_{c,d}(\alpha)\,, \end{aligned}$$

which proves Theorem 6.

Theorem 7. Let $\mu > 0$ and $0 \le \alpha, \beta < 1$. *If* $f(z) \in K^{s,a}_{c,d}(\beta, \alpha)$, *then*

$$J_\mu(f)(z) \in \Sigma K^{s,a}_{c,d}(\beta, \alpha)\,.$$

Suppose that $f(z) \in \Sigma K^{s,a}_{c,d}(\beta, \alpha)$. Then, by Definition 3, there exists a function $g(z) \in \Sigma C^{s,a}_{c,d}(\beta, \alpha)$ such that

$$Re\left(\frac{z\left({}^{s}_{d}(a,c;z)f(z)\right)^{'}}{{}^{s}_{d}(a,c;z)g(z)}\right) < -\beta \quad (z \in U)\,.$$

Thus, upon setting

$$\frac{z\left({}^{s}_{d}(a,c;z)J_\mu f(z)\right)^{'}}{{}^{s}_{d}(a,c;z)J_\mu g(z)} + \beta = -(1-\beta)q(z)\,, \tag{3.8}$$

where $q(z)$ is given by (1.14), we find from (3.2) that

$$\begin{aligned} \frac{z\left({}^{s}_{d}(a,c;z)f(z)\right)^{'}}{{}^{s}_{d}(a,c;z)g(z)} &= -\frac{{}^{s}_{d}(a,c;z)(-zf^{'}(z))}{{}^{s}_{d}(a,c;z)g(z)} \\ &= -\frac{(\mu+1){}^{s}_{d}(a,c;z)J_\mu(-zf^{'}(z)) + z({}^{s}_{d}(a,c;z)J_\mu(-zf^{'}(z)))^{'}}{(\mu+1){}^{s}_{d}(a,c;z)J_\mu g(z) + z({}^{s}_{d}(a,c;z)J_\mu(g(z)))^{'}} \\ &= -\frac{\frac{z({}^{s}_{d}(a,c;z)J_\mu(-zf^{'}(z)))^{'}}{{}^{s}_{d}(a,c;z)J_\mu g(z)} + (\mu+1)\frac{{}^{s}_{d}(a,c;z)J_\mu(-zf^{'}(z))}{{}^{s}_{d}(a,c;z)J_\mu g(z)}}{\frac{z({}^{s}_{d}(a,c;z)J_\mu g(z))^{'}}{{}^{s}_{d}(a,c;z)J_\mu g(z)} + (\mu+1)}\,. \end{aligned}$$

Since $g(z) \in S^{s,a}_{c,d}(\alpha)$, we know from Theorem 5 that $J_\mu g(z) \in S^{s,a}_{c,d}(\alpha)$. So we can set

$$\frac{z({}^{s}_{d}(a,c;z)J_\mu g(z))^{'}}{{}^{s}_{d}(a,c;z)J_\mu g(z)} + \alpha = -(1-\alpha)G(z)\,, \tag{3.9}$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \text{ and } Re(G(z)) = g_1(x,y) > 0 \quad (z \in U)\,.$$

Then we have

$$\frac{z({}^{s}_{d}(a,c;z)f(z))'}{{}^{s}_{d}(a,c;z)g(z)} = \frac{\frac{z(I^{m}_{p,\mu}(\lambda,\ell)J_{c,p}(-zf'(z)))'}{I^{m}_{p,\mu}(\lambda,\ell)J_{c,p}g(z)} + (\mu+1)\left[\beta+(1-\beta)q(z)\right]}{\alpha+(1-\alpha)G(z)-(\mu+1)} . \qquad (3.10)$$

We also find from (3.8) that

$$z({}^{s}_{d}(a,c;z)J_{\mu}f(z))' = (-{}^{s}_{d}(a,c;z)J_{\mu}g(z))\left[\beta+(1-\beta)q(z)\right] . \qquad (3.11)$$

Differentiating both sides of (3.11) with respect to z, we obtain

$$\begin{aligned} z\left[z\left(I^{m}_{p,\mu}(\lambda,\ell)J_{\mu}f(z)\right)'\right]' &= -z({}^{s}_{d}(a,c;z)J_{\mu}g(z))'\left[\beta+(1-\beta)q(z)\right] \\ &\quad -(1-\beta)zq'(z){}^{s}_{d}(a,c;z)J_{\mu}g(z)\,, 3.12 \qquad (3) \end{aligned}$$

that is,

$$\begin{aligned} \frac{z\left[z\left({}^{s}_{d}(a,c;z)J_{\mu}f(z)\right)'\right]'}{{}^{s}_{d}(a,c;z)J_{\mu}g(z)} &= -(1-\beta)zq'(z) + \\ &\quad \left[\alpha+(1-\alpha)G(z)\right]\left[\beta+(1-\beta)q(z)\right] .3.13 \qquad (4) \end{aligned}$$

Substituting (3.13) into (3.10), we find that

$$\frac{z\left({}^{s}_{d}(a,c;z)f(z)\right)'}{{}^{s}_{d}(a,c;z)g(z)} + \beta = -(1-\beta)q(z) + \frac{(1-\beta)zq'(z)}{(1-\alpha)G(z)+\alpha-(\mu+1)} . \qquad (3.14)$$

Then, by setting

$$u = q(z) = u_1 + iu_2 \text{ and } v = zq'(z) = v_1 + iv_2 ,$$

we can define the function $\theta(u, v)$ by

$$\theta(u,v) = (1-\beta)u - \frac{(1-\beta)v}{(1-\alpha)G(z)+\alpha-(\mu+1)} .$$

The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved.

Theorem 8. *Let* $\mu > 0$ and $0 \le \alpha, \beta < 1$. *If* $f(z) \in K^{*s,a}_{c,d}(\beta,\alpha)$, *then*

$$J_{\mu}(f)(z) \in K^{*s,a}_{c,d}(\beta,\alpha) .$$

Just as we derived Theorem 6 from Theorem 5, we easily deduce Theorem 8 from Theorem 7.

Application

By specifying the parameters d, s, a and c we obtain various results for different operators.

References

[1] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, Certain integral operators applied to meromorphic p-valent functions, J. Nat. Geom., 24(2003), 111-120.

[2] S. K. Bajpai, A note on a classes of meromorphic univalent functions, Revue Roum. Math. Pures Appl., (1977), no. 22, 295-297.

[3] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations, Integral Transforms Special Functions, 16(2005), no. 18, 647-659.

[4] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta functions. Appl. Math. Comput., 170(2005), 399-409.

[5] C. Ferreira and J. L. Lopez, Asymptotic expansionsof the Hurwitz-Lerch Zeta function. Journal of Math. Anal. Appl., 298(2004), 210-224.

[6] R. M. Goel and N. S. Sohi, On a class of meromophic functions, Glasnik Math. III, 17(1982), no. 37, 19-28.

[7] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc., 2(1971), no. 3, 469-474.

[8] A. Y. Lashin, On certain subclass of meromorphic functions associated with certain integral operators, Comput. Math Appl., 59(2010), no.1, 524-531.

[9] R. J. Libera and M. S. Robertson, Meromorphic close-to-convex functions, Michigan Math. J., (1961), no. 8, 167-176.

[10] S.-D. Lin and H.M Srivastava, Some families of the Hurwitz-Lerrch Zeta functions and associated fractional dervtives and other integral representations, Appl. Math. Comput., 154(2004), 725-733.

[11] Q.-M. Luo and H. M. Srivastava, Some generalization of the Apostol-Bernoulli and Apostol-Euler polynomials, J. Math. Anal. Appl., 308(2005), 290-302.

[12] S. S. Miller and P. T. Mocanu, Differential Subordinations : Theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math. No. 225 Marcel Dekker, Inc. New York, 2000.

[13] S. S. Miller and P. T. Mocanu, Second-order differential inequalities in the complex plane, J. Math. Anal. Appl., 65(1978), 289-305.

[14] R. Singh, Meromorphic close-to-convex functions, J. Indian Math. Soc., 33(1969), 13-20.

[15] H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[16] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions (Dordrecht, Boston and London: Kluwer Academic Publishers), 2001.

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