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A PROOF THAT S-UNIMODAL MAPS ARE COLLET-ECKMANN MAPS IN A SPECIFIC RANGE OF THEIR BIFURCATION PARAMETERS

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Abstract. Generally, Collet-Eckmann maps require unimodality and multimodality. The inverse is not true. In this paper, we will prove that S-unimodal maps are Collet-Eckmann maps in a specific range of their bifurcation parameters. The proof is based on the fact that the family of robustly chaotic unimodal maps known in the literature are all topologically conjugate to one another and the fact that if two S-unimodal maps of the interval are conjugate by a homeomorphism of the interval and if one of them is Collet-Eckmann, then so is the other one.

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1.Introduction

Definitions:

- We say that c is a *nonflat critical point* of a map f of the interval if $f'(c)=0$ but for some $l_c>1$, the limit $\lim_{x\to+c}\frac{|f'(x)|}{|x-c|^{l_c-1}}$ exists and is nonzero.
- A C^2 map f of the interval is called S*-multimodal* if (i) f has a finite number of nonflat critical points, and (ii) $|f'|^{-1/2}$ is convex between the critical points.
- A map $f: I=[a,b]\longrightarrow I$ is S*-unimodal* on the interval I if (a) the function $f(x)$ is of class C^3, (b) the point a is a fixed point with b its other preimage, i.e., $f(a)=f(b)=a$, (c) there is a unique maximum at $c\in(a,b)$ such that $\varphi(x)$ is strictly increasing on $x\in[a,c)$ and strictly decreasing on $x\in(c,b]$, and (d) the function f has a negative Schwarzian derivative, i.e., $S(f,x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2<0$ for all $x\in I-\{y,f'(y)=0\}$.

- An S-multimodal map f is called Collet-Eckmann (CE) if there exist constants $C > 0$ and $\lambda > 1$ such that for every $n > 0$ and every critical point c ($f'(c) = 0$) whose forward trajectory does not meet other critical points, we have that $\left|(f^n)'(f(c))\right| > C\lambda^n$. In this case, the S-multimodal Collet-Eckmann maps are strongly hyperbolic along the critical orbit.

Robust chaos [5] is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space. See [7] for more details. The existence of such windows implies that small changes of the parameters would destroy the chaos, implying the fragility of this type of chaos. We note that S-unimodal maps are very important in chaos theory according to the theorem given in [13]. This theorem claims that every attracting periodic orbit attracts at least one critical point or boundary point. Hence an S-unimodal map can have at most one periodic attractor which will attract the critical point. Let $f(x) : I \to I$ be an S-unimodal map on the interval $I = [a, b]$. Then every attracting periodic orbit attracts at least one critical point or boundary point. Furthermore, each neutral periodic orbit is attracting. The following important result can be derived from this theorem: An S-unimodal map can have at most one periodic attractor which will attract the critical point.

Collet-Eckmann maps have the following properties: (a) They are characterized by a positive Lyapunov exponent for the critical value. (b) They are the *best possible* near hyperbolic properties [14-15], for example, exponential decay of correlations, existence of central limit and large deviations theorems, good spectral properties, and zeta functions. (c) They have a robust statistical description, strong stochastic stability [16], and rates of convergence to equilibrium [17].

Proving that a map is Collet-Eckmann is not an easy task because the main inequality $\left|(f^n)'(f(c))\right| > C\lambda^n$ requires information or expression of the n^{th} composition of the map f for all $n > 0$. In fact, there are some proofs in the literature about finding conditions for an S-unimodal map to be a Collet-Eckmann map. For example, it was shown in [12] that an S-unimodal map satisfies the Collet-Eckmann condition if and only if it has a uniform hyperbolic structure.

2. Main results for the proof

In this section, we will state three main results concerning our proof. Firstly, the following result was shown in [18]: If F and G are S-unimodal maps of the interval conjugated by a homeomorphism h of the interval, and F is Collet-Eckmann, then so is G.

Secondly, the following result was shown in [1]: the map $f_\nu(x) = \frac{1+\nu-\nu^x-\nu^{1-x}}{\left(\sqrt{\nu}-1\right)^2}, \nu > 0, \nu \neq 1, x \in [0, 1]$ is an S-unimodal map on the interval $[0, 1]$.

Thirdly, it was shown in [11] that if $\{f_t\}$ is a smooth unimodal family with robust chaos, then all maps within this family are topologically conjugate. This means that the family of robustly chaotic unimodal maps known in the literature are all topologically conjugate to one another.

In this paper we will prove the existence of a subinterval H in the space of bifurcation parameters such that any S-unimodal map on the interval $I = [0, 1]$ is a Collet-Eckmann map.

The method of analysis is based on the following three steps:

Step 1: Firstly, we prove the property for one known S-unimodal map: $f_\nu(x) = \frac{1+\nu-\nu^x-\nu^{1-x}}{\left(\sqrt{\nu}-1\right)^2}, \nu > 0, \nu \neq 1$, and $x \in [0, 1]$ given in [1].

Step 2: Secondly, we use the fact that all other S-unimodal maps are topologically conjugate to f_ν [11].

Step 3: Thirdly, we use the fact that if F and G are S-unimodal maps of the interval conjugated by a homeomorphism h of the interval and F is Collet-Eckmann, then so is G [18].

Proof of **Step 1**: We will show that $f_\nu(x) = \frac{1+\nu-\nu^x-\nu^{1-x}}{\left(\sqrt{\nu}-1\right)^2}, \nu > 0, \nu \neq 1, x \in [0, 1]$ is a Collet-Eckmann map. Indeed, it was shown in [1] that f_ν is S-unimodal. We have $f_\nu(0) = f_\nu(1) = 0, f_\nu(x)$ is a smooth map, $\forall x \in [0, 1], \forall \nu > 0, \nu \neq 1$, also, $c = \frac{1}{2},, f'_\nu(c) = 0, f_\nu(c) = 1, \forall \nu > 0, \nu \neq 1$. Also, $\lim_{\nu \to 0^+} f_\nu(x) = 1, \forall x \in [0, 1]$ and $f_\nu(x)$ has negative Schwarzian derivative $(x \neq c, \nu \neq 1)$: $S(f_\nu(x), x) = -[\ln(\nu)]^2 \left[1 + \frac{3}{2}\left(\frac{\nu^x+\nu^{1-x}}{\nu^x-\nu^{1-x}}\right)^2\right] < 0$. Hence, f_ν is an S-unimodal map. Note that it is possible to restrict the study of the map f_ν to the interval $\nu \in (0, 1)$, because $f_\nu(x) = f_{1/\nu}(x), \forall \nu > 0,\ \nu \neq 1$.

On the other hand, we have $f^n_\nu(x) = \frac{1}{\left(\sqrt{\nu}-1\right)^2}\left(1 + \nu - \nu^{f^{n-1}_\nu(x)} - \nu^{1-f^{n-1}_\nu(x)}\right)$. Thus, $(f^n_\nu(x))' = \frac{-\left(f^{n-1}_\nu(x)\right)' \ln v}{\left(\sqrt{\nu}-1\right)^2}\left(v^{f^{n-1}_\nu(x)} - \nu^{1-f^{n-1}_\nu(x)}\right)$. We have $c = \frac{1}{2}, f_\nu(c) = 1, \forall \nu > 0, \nu \neq 1$. Thus $\left|(f^n_\nu)'(f(c))\right| = \left|(f^n_\nu)'(1)\right| = \left|\frac{-\left(f^{n-1}_\nu(1)\right)' \ln v}{\left(\sqrt{\nu}-1\right)^2}\left(v^{f^{n-1}_\nu(1)} - \nu^{1-f^{n-1}_\nu(1)}\right)\right|$. We have $f_\nu(0) = f_\nu(1) = 0$, then $f^2_\nu(1) = f_\nu(f_\nu(1)) = f_\nu(0) = 0$, therefore by induction we can prove that $f^n_\nu(1) = 0$ for all $n \geq 0$. Hence, $\left|(f^n_\nu)'(1)\right| = \left|\left(\frac{-(1-\nu)\ln v}{\left(\sqrt{\nu}-1\right)^2}\right)\left(f^{n-1}_\nu(1)\right)'\right|$. By setting $\alpha_n = \left|(f^n_\nu)'(1)\right|, n \geq 1$, we obtain $\alpha_n = \beta\alpha_{n-1}$, $n \geq 1$, where $\beta = \frac{-(1-\nu)\ln v}{\left(\sqrt{\nu}-1\right)^2} > 0$. Hence, $\alpha_n = \beta^{n-1}\alpha_1$ where $\alpha_1 = \left|(f_\nu)'(1)\right| = \frac{-(1-\nu)(\ln \nu)}{\left(\sqrt{\nu}-1\right)^2} = \beta$. Thus, $\left|(f^n_\nu)'(1)\right| = \beta^n.1 > \beta^n.C$ for any $C \in (0, 1)$. Let $\lambda = \beta$. We will prove that $\lambda > 1$ for some $\nu \in (0, 1)$. Indeed, $\lambda > 1 \Leftrightarrow \ln\frac{1}{v} > \frac{\left(\sqrt{\nu}-1\right)^2}{(1-\nu)}$, hence let us consider the function $\varphi(v) = \ln\frac{1}{v} - \frac{\left(\sqrt{v}-1\right)^2}{(1-v)}$ for $\nu \in (0,1)$. Thus, $\lambda > 1 \Leftrightarrow \varphi(v) > 1$. We have $\varphi'(v) = \frac{-\left(v+\sqrt{v}+1\right)}{v\left(\sqrt{v}+1\right)^2} < 0$ for $\nu \in (0,1)$. Thus the function φ is decreasing. Hence $\varphi(v) > \varphi(v')$ for all $v < v'$. When $v' \to 1$, we have $\varphi(v) > \varphi(1) = \lim_{v' \to 1} \varphi(v) = 0$. Finally, we have $\varphi(v) > 0$ for all $\nu \in (0,1)$.

The second and the third steps imply that all S-unimodal maps on the interval $[0,1]$ are Collet-Eckmann for all $\nu \in (0,1)$. Now, it is well known that any interval $[a,b]$ is homeomorphic to the interval $[0,1]$. Thus, the above analysis (in the interval $x \in [0,1]$) holds true for any interval $[a,b]$.

Thus we have proved that any S-unimodal map is a Collet-Eckmann map in a specific range of its bifurcation parameter.

3.Conclusion

In this paper, we have proved that all S-unimodal maps are Collet-Eckmann maps in a specific range of their bifurcation parameters. The proof is based on the fact that the family of robustly chaotic unimodal maps known in the literature are all topologically conjugate to one another and the fact that if two S-unimodal maps of the interval are conjugate by a homeomorphism of the interval and if one of them is Collet-Eckmann, then so is the other one.

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