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ON THE STABILITY OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION

AKKOUCHI Mohamed and ED-DARRAZ Abdelkarim

Abstract. Let X be a real linear space, Y be a real sequentially complete Hausdorff topological vector space and consider the following generalized quadratic functionl equation:

$$f(ax+by)+f(ax-by)=c[a^2f(x)+b^2f(y)], \quad \forall x,y\in X, \qquad (E-Q)(a,b,c)$$

where a,b and c are real parameters and $f:X\to Y$ is unknown. The aim of this paper is to study the Hyers-Ulam and Hyers-Ulam-Rassias-Aoki-Gavruta stability of the equation $(E-Q)(a,b,c)$. Our results generalize those obtained quite recently by M. Kumar and A. Kumar in the paper [*Hyers-Ulam-Rassias stability of generalized quadratic functional equations*, International J. of Math. Archive. 3 (2), Feb., (2012), 485-490]. Our paper generalizes and unifies the results of several other papers devoted to the stability of quadratic functionals. We present also a fixed point method to treat the stability of Equation $(E-Q)(a,b,c)$ based on the Banach contraction priniciple without making appeal to its alternative established by J. B. Diaz and B. Margolis in 1968. So our method is different from the one adopted in several recent papers treating stability of functional equations by the alternative fixed point method.

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1. INTRODUCTION

In 1940, S.M.Ulam (see [22] and [23]) introduced the notion of stability of a homomorphism as follows:

Let $(G_1,+)$ be a group and $(G_2,*,d)$ a metric group. Given $\varepsilon>0$, dose there exists a $\delta>0$ such that if $f:G_1\longrightarrow G_2$ satisfies

$$d(f(x+y),f(x)*f(y))\leq\delta, \quad \forall x,y\in G_1,$$

then there exists a homomorphism $g : G_1 \longrightarrow G_2$ with

$$d(f(x), g(x)) \leq \varepsilon, \quad \forall x \in G_1.$$

One year later, D. H. Hyers gives an affirmative answer in the case where G_2 is a Banach space (see [8]). The stability problems of a various functional equations have been extensively investigated and studied by number of authors (see [1], [6], [9], [10], [12,13], [14],[17] [18], [19]).

In [15] D. Popa was proved a similre Hyers-Ulam stability of linear equation for a mapping from a real linear space X into a real sequentially complete Hausdorff topological vector space Y such that

$$f(ax + by + k) - pf(x) - qf(y) - s \in V,$$

where V is a subset of Y

In this work, we will treat the stability of the following generalized quadratic equation:

$$f(ax + by) + f(ax - by) = c[a^2 f(x) + b^2 f(y)], \quad \forall x, y \in X, \qquad (E-Q)(a, b, c).$$

where a, b and c are real parameters and $f : X \to Y$ is unknown from the linear space X to a sequentially complete Hausdorff topological vector space Y. We study the stability of Equation $(E - Q)(a, b, c)$ in the sense of Hyers-Ulam and in the sense of Hyers-Ulam-Rassais-Aoki-Gavruta. Also, we give a fixed point method to establish the stability of this equation We point out that our fixed point method is based only on the Banach fixed point theorem and that we do not make use of its alternative as it is usually done in several recent papers dealing with stability of functional equations by means of fixed point theorems (see [2], [5], [16], [20]). We point out, also, that the main theorems in [11] will be obtained as application of our results.

Let X be a linear space and Y be a sequentially complete Hausdorff topological vector space over the real field. Let $B : X \times X \to Y$ be a bilinera form and set $f(x) = B(x, x)$ for all $x \in X$. Then is easy to see that f is a solution of the equation $(E - Q)(a, b, 2)$, for all real numbers a, b. This explains why the equations $(E - Q)(a, b, c)$ are called generalized quadratic equations.

F. Skof was the first proved in [21] the hyers-Ulam stability of the quadratic functional equation $(E-Q)(a, b, c)$, so P. W. Cholewa [3] extend Skof reslut to a abelian group. In [4] S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation and this result was generalized by Grabiec (see [7]).

In all this work we suppose that a, b, $c \in \mathbb{R}^*$ with $a \neq \pm 1$, and we consider the numbers

$$r_1 = \frac{2}{ca^2}, \ r_2 = r_1^{-1}, \ a_1 = a \text{ and } a_2 = a_1^{-1}.$$

In section 2 and 3 we the stability of the equation $(E-Q)(a,b,c)$ in the sense of Hyers-Ulam and in the sense of Hyers-Ulam-Rassais-Aoki-Gavruta using the direct knowing method. The fourth section give the stability of the same equation but using the fixed point method, so the proof will be more concentrate.

2. HYERS-ULAM STABILITY OF $(E-Q)(a,b,c)$

In this section we will study the stability in the sense of Hyers-Ulam of the equation $(E-Q)(a,b,c)$, where the function f is defined from the linear space X into a real sequentially complete Hausdorff topological vector space Y.
We set

$$\Delta_{a,b,c}f(x,y) = f(ax+by) + f(ax-by) - c[a^2 f(x) + b^2 f(y)], \quad \forall x, y \in X.$$

Our first result it's as follows

Theorem 1. *Let X be a linear space, Y a sequentially complete Hausdorff topological vector space over $\mathbb{R}$ and $V \subseteq Y$ is a nonempty bounded, convex and symetric set.*
Let $f : X \longrightarrow Y$ be a application such that $f(0) = 0$ and :

$$\Delta_{a,b,c}f(x,y) \in V, \quad \forall x,\, y \in X \qquad (1).$$

Suppose that $|r_1| < 1$ (resp. $|r_2| < 1$). Then, there exists a unique application $g^1 : X \longrightarrow Y$ (resp. $g^1 : X \longrightarrow Y$) which satisfies $(E-Q)(a,b,c)$, $g^1(0) = 0$ (resp.$g^2(0) = 0$) and

$$g^1(x) - f(x) \in \frac{|r_1|}{2-2|r_1|}.[V]_{seq}, \quad \forall x \in X \qquad (Approx(1.1)).$$

resp.

$$g^2(x) - f(x) \in \frac{1}{2-2|r_2|}.[V]_{seq}, \quad \forall x \in X \qquad (Approx(1.2)).$$

Where $[V]_{seq}$ denot the sequential closure of V.

Proof Taking $y = 0$ in (1), for all $x \in X$ we have

$$2f(ax) - ca^2 f(x) \in V.$$

Since, for all $x \in X$ we have

$$r_1 f(a_1 x) - f(x) \in \frac{r_1}{2}.V$$

Let $x \in X$ and $n \in \mathbb{N}$, by symmetry of V, we have

$$r_1 f(a_1^{n+1}x) - f(a_1^n x) \in \frac{|r_1|}{2}.V,$$

resp.

$$f(a_2^{n+1}) - r_1 f(a_2^n x) \in \frac{|r_1|}{2}.V.$$

For all $n \in \mathbb{N}$ and $x \in X$, let us consider

$$f_n^1(x) = r_1^n f(a_1^n x) \text{ (resp. } f_n^2(x) = r_2^n f(a_2^n x) \text{).}$$

So

$$f_{n+1}^1(x) - f_n^1(x) \in \frac{|r_1|}{2}.|r_1|^n.V, \quad \forall x \in X, \ \forall n \in \mathbb{N},$$

resp.

$$f_{n+1}^2(x) - f_n^2(x) \in \frac{|r_1|}{2}.|r_2|^{n+1}.V, \quad \forall x \in X, \ \forall n \in \mathbb{N}.$$

Let $x \in X$ and $m \geq n \in \mathbb{N}$, by convexity of V, we have

$$\begin{aligned} f_m^1(x) - f_n^1(x) &= \sum_{k=n}^{m-1} \{f_{k+1}^1(x) - f_k^1(x)\} \\ &\in \frac{|r_1|}{2}. \sum_{k=n}^{m-1} |r_1|^k.V \\ &\subseteq \frac{|r_1|}{2}.\{\sum_{k=n}^{m-1} |r_1|^k\}.V \end{aligned}$$

resp.

$$\begin{aligned} f_m^2(x) - f_n^2(x) &= \sum_{k=n}^{m-1} \{f_{k+1}^1(x) - f_k^1(x)\} \\ &\in \frac{|r_1|}{2}. \sum_{k=n}^{m-1} |r_2|^{k+1}.V \\ &\subseteq \frac{|r_1|}{2}.\{\sum_{k=n}^{m-1} |r_2|^{k+1}\}.V \end{aligned}$$

Let $x \in X$ and U be an arbitrary neighbourhood of $0 \in Y$. Since V is bounded, there exists $\lambda > 0$ such that $\frac{|r_1|}{2}.\lambda.V \subseteq U$.

Let $N_0 \in \mathbb{N}$ such that,

$$\sum_{k=n}^{m-1} |r_1|^k < \lambda \text{ (resp. } \sum_{k=n+1}^{m} |r_2|^k < \lambda \text{).}$$

for every $m > n \geq N_0 \in \mathbb{N}$. Then for $m > n \geq N_0 \in \mathbb{N}$

$$f_m^i(x) - f_n^i(x) \in \frac{|r_1|}{2}.\lambda.V \subseteq U, \quad i = 1, 2.$$

Then, for each $i = 1, 2$ and all $x \in X$, the sequence $(f_n^i(x))_{n\in\mathbb{N}}$ is a Cauchy sequence on Y.
Let consider $g^1 : X \longrightarrow Y$ (resp. $g^2 : X \longrightarrow Y$) defined for all $x \in X$ by

$$g^1(x) = \lim_{n\longrightarrow+\infty} f_n^1(x) \text{ (resp. } g^2(x) = \lim_{n\longrightarrow+\infty} f_n^2(x) \text{).}$$

For each $i = 1, 2$, $x \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} f_n^i(x) - f(x) &= f_n^i(x) - f_0^i(x) \\ &\in \frac{|r_1|}{2}.\begin{cases} \{\sum_{k=0}^{n-1} |r_1|^k\}.V \\ \{\sum_{k=0}^{n-1} |r_2|^{k+1}\}.V \end{cases} \end{aligned}$$

So, for all $x \in X$ we have

$$g^1(x) - f(x) \in \frac{|r_1|}{2-2|r_1|}.[V]_{seq} \text{ (resp. } g^2(x) - f(x) \in \frac{1}{2-2|r_2|}.[V]_{seq} \text{).}$$

For each $i = 1, 2$, remplacing (x, y) by $(a_i^n x, a_i^n y)$ and multipled by r_i^n, $n \in \mathbb{N}$ the relation (1) leads to

$$\Delta_{a,b,c} f_n^i(x, y) \in |r_i|^n.V, \quad \forall x, y \in X.$$

So, for each $i = 1, 2$ we have $\Delta_{a,b,c} g^i = 0$ and $g^i(0) = 0$.
Let $h^1 : X \longrightarrow Y$ (resp. $h^2 : X \longrightarrow Y$) be a application which is solution of $(E - Q)(a, b, c)$, $h^1(0) = 0$ (resp. $h^2(0) = 0$) and satisfies $(Approx(1.1))$ (resp. $(Approx(1.2))$). Then, for each $i = 1, 2$ we have $h^i = g^i$:

$$\begin{aligned} h^i(x) - g^i(x) &= r_i^n\{h^i(a_i^n x) - g^i(a_i^n x)\} \\ &= r_i^n\{h^i(a_i^n x) - f(a_i^n x)\} + r_i^n\{f(a_i^n x) - g^i(a_i^n x)\} \\ &\in \begin{cases} |r_1|^n.\dfrac{|r_1|}{1-|r_1|}.[V]_{seq} \\ |r_2|^n.\dfrac{1}{1-|r_2|}.[V]_{seq} \end{cases} \end{aligned}$$

for each $i = 1, 2$*, all* $x \in X$ *and* $n \in \mathbb{N}$*, so by taking limits we conclud that* $h^1 = g^1$ *(resp.* $g^2 = h^2$*).*

3. THE HYERS-ULAM-RASSIAS-AOKI-GAVRUTA STABILITY OF $(E - Q)(a, b, c)$

In this section we will study the stability in the sense of Hyers-Ulam-Rassis-Aoki-Gavruta of the equation $(E - Q)(a, b, c)$, where the function f is defined from a linear space X into a sequentially complete Hausdorff topological vector space over $\mathbb{R}$.

Let $\varphi : X \times X \longrightarrow \mathbb{R}^+$ be a control function and let $\eta : X \longrightarrow \mathbb{R}^+$ be the fonction defined by

$$\eta(x) = \varphi(x, 0), \quad \forall x \in X.$$

Theorem 2. *Let* X *be a linear space,* Y *be a sequentially complete Hausdorff topological vector space over* $\mathbb{R}$ *and* V *is a nonempty bounded, convex and symetric set of* Y*.*

Let $f : X \longrightarrow Y$ *be a applicationis such that* $f(0) = 0$ *and :*

$$\Delta_{a,b,c} f(x, y) \in \varphi(x, y).V, \quad \forall x,\, y \in X \qquad (2).$$

Suppos that

$$\eta_1(x) = \sum_{n \geq 0} |r_1|^n \eta(a_1^n x) < \infty,\ \forall x \in X \ \textit{(resp.}\ \eta_2(x) = \sum_{n \geq 1} |r_2|^n \eta(a_2^n x) < \infty,\ \forall x \in X\ \textit{)},$$

and

$$\lim_{n \longrightarrow +\infty} |r_1|^n \varphi(a_1^n x, a_1^n y) = 0 \ \textit{(resp.} \lim_{n \longrightarrow +\infty} |r_2|^n \varphi(a_2^n x, a_2^n y) = 0\ \textit{)}.$$

Then, there exists a unique application $g^1 : X \longrightarrow Y$ *(resp.* $g^2 : X \longrightarrow Y$*) which is solution of* $(E - Q)(a, b, c)$ *such that* $g^1(0) = 0$ *(resp.* $g^2(0) = 0$*) and satisfies*

$$g^1(x) - f(x) \in \frac{|r_1|}{2} \eta_1(x).[V]_{seq}, \quad \forall x \in X \qquad (Approx(2.1)),$$

resp.

$$g^2(x) - f(x) \in \frac{|r_1|}{2} \eta_2(x).[V]_{seq}, \quad \forall x \in X \qquad (Approx(2.2)).$$

Proof. *For* $y = 0$ *in* (2) *and by symmetry of* V *we have*

$$r_1 f(a_1 x) - f(x) \in \frac{|r_1|}{2}.\eta(x).V, \quad \forall x \in X.$$

Hence, for all $x \in X$ *and* $n \in \mathbb{N}$*, we have*

$$f^1_{n+1}(x) - f^1_n(x) \in \frac{|r_1|}{2}.|r_1|^n.\eta(a_1^n x).V$$

resp. (by symmetry of V *)*

$$f^2_{n+1}(x) - f^2_n(x) \in \frac{|r_1|}{2}.|r_2|^{n+1}.\eta(a_2^{n+1} x).V$$

Where, for each $i = 1, 2$*, all* $x \in X$ *and* $n \in \mathbb{N}$

$$f^i_n(x) = r_i^n f(a_i^n x).$$

Let $x \in X$ *and* $m \geq n \in \mathbb{N}$*, by convexity of* V*, for each* $i = 1, 2$ *we have*

$$\begin{aligned} f^i_m(x) - f^i_n(x) &= \sum_{k=n}^{m-1} \{f^i_{k+1}(x) - f^i_k(x)\} \\ &\in \frac{|r_1|}{2}.\begin{cases} \{\sum_{k=n}^{m-1} |r_1|^k \eta(a_1^k x)\}.V \\ \{\sum_{k=n+1}^{m} |r_2|^k \eta(a_2^k x)\}.V \end{cases} \end{aligned}$$

Now, using the fact that V *is bounded and the condtion on* η_1 *(resp.* η_2*) we show that; for each* $i = 1, 2$ *and all* $x \in X$*, the sequence* $(f^i_n(x))_{n\in\mathbb{N}}$ *is a Cauchy sequence on* Y*, so converge.*
Let $g^1 : X \longrightarrow Y$ *(resp.* $g^2 : X \longrightarrow Y$ *) be the application defined by*

$$g^1(x) = \lim_{n\longrightarrow+\infty} f^1_n(x) \ (resp.\ g^2(x) = \lim_{n\longrightarrow+\infty} f^2_n(x)\ ).$$

for all $x \in X$*. For each* $i = 1, 2$*, all* $x \in X$ *and* $n \in \mathbb{N}$ *we have*

$$f^i_n(x) - f(x) \in \frac{|r_1|}{2}.\begin{cases} \{\sum_{k=0}^{n-1} |r_1|^k \eta(a_1^n k)\}.V \\ \{\sum_{k=1}^{n} |r_2|^k \eta(a_2^n k)\}.V \end{cases}$$

So, for all $x \in X$ *we have*

$$g^1(x) - f(x) \in \frac{|r_1|}{2}.\eta_1(x).[V]_{seq} \ (resp.\ g^2(x) - f(x) \in \frac{|r_1|}{2}.\eta_2(x).[V]_{seq}\ ).$$

For each $i = 1, 2$, *remplacing* (x, y) *by* $(a_i^n x, a_i^n y)$ *and multipled by* r_i^n, $n \in \mathbb{N}$ *the relation* (1) *leads to*

$$\Delta_{a,b,c} f_n^i(x, y) \in |r_i|^n \varphi(a_i^n x, a_i^n y).V, \quad \forall x, y \in X.$$

So, for each $i = 1, 2$ *we have* $\Delta_{a,b,c} g^i = 0$ *and* $g^i(0) = 0$.
Let $h^1 : X \longrightarrow Y$ *(resp.* $h^2 : X \longrightarrow Y$ *) be a other application which is solution of* $(E - Q)(a, b, c)$, $h^1(0) = 0$ *(resp.* $h^2(0) = 0$ *) and satisfies* $(Approx(2.1))$ *(resp.* $(Approx(2.2))$*), for each* $i = 1, 2$, *all* $x \in X$ *and* $n \in \mathbb{N}$ *we have*

$$\begin{aligned} h^i(x) - g^i(x) &= r_i^n\{h(a_i^n x) - g(a_i^n x)\} \\ &= r_i^n\{h(a_i^n x) - f(a_i^n x)\} + r_i^n\{f(a_i^n x) - g(a_i^n x)\} \\ &\in |r_1|.|r_i|^n.\eta_i(a_i^n x).[V]_{seq} \end{aligned}$$

By taking the limits, we show that $h^1 = g^1$ *(resp.* $h^2 = g^2$*).*

4. STABILITY BY FIXED POINT METHOD OF $(E - Q)(a, b, c)$

By using the Banach principle, we well study the Hyers-Ulam and the Hyers-Ulam-Rassias -Aoki-Gavruta stability of the functional equation $(E - Q)(a, b, c)$.
Theorem 3. *Let* X *be a linear space,* Y *be a sequentially complete Hausdorff topological vector space over* $\mathbb{R}$ *and* V *a nonempty bounded, convex and symetric set of* Y*:* $[[V]_{seq}]_{seq} = [V]_{seq}$.
Let $f : X \longrightarrow Y$ *be a application such that* $f(0) = 0$ *satisfies* (1) *and suppos that* $|r_1| < 1$ *(resp.* $|r_2| < 1$*).*
Then, there exists a unique application $g^1 : X \longrightarrow Y$ *(resp.* $g^2 : X \longrightarrow Y$ *) which is solution of* $(E - Q)(a, b, c)$ *such that* $g^1(0) = 0$ *(resp.* $g^2(0) = 0$*) and satisfies* $(Approx(1.1))$ *(resp.* $(Approx(1.2))$*).*
Proof Taking in (1) $y = 0$, *for all* $x \in X$, *by symmetry of* V, *we have*

$$r_1 f(a_1 x) - f(x) \in \frac{|r_1|}{2}.V,$$

resp.

$$r_2 f(a_2 x) - f(x) \in \frac{1}{2}.V.$$

Consider

$$E_f := \{g : X \longrightarrow Y;\ \exists M = M_g \in \mathbb{R}^+ :\ g(x) - f(x) \in M.[V]_{seq}, \quad \forall x \in X\},$$

and define the metric δ *on* E_f *by*

$$\delta(g, h) := \inf\{M \in \mathbb{R}^+;\ g(x) - h(x) \in M.[V]_{seq}\}, \quad \forall g, h \in E_f.$$

We can show that (E_f, δ) *is a complete metric space.*
For each $i = 1, 2$, *we define*

$$T_i : E_f \longrightarrow E_f,\ g \longmapsto T_i g(x) = r_i g(a_i x), \quad \forall \in X,$$

for each $i = 1, 2$, *the operators* T_i *is well defined : Let* $g \in E_f$ *and* $x \in X$, *using the convexity of* $[V]_{seq}$, *we have*

$$\begin{array}{rcl} T_1 g(x) - f(x) & = & T_1(g - f)(x) + T_1 f(x) - f(x) \\ & \in & \{M + \dfrac{1}{2}\}|r_1|.[V]_{seq} \end{array}$$

resp

$$\begin{array}{rcl} T_2 g(x) - f(x) & = & T_2(g - f)(x) + T_2 f(x) - f(x) \\ & \in & \{|r_2|M + \dfrac{1}{2}\}.[V]_{seq} \end{array}$$

Let $(g, h) \in (E_f \times E_f)$, $M \in \mathbb{R}^+$ *such that* $\delta(g, h) < M$, *for all* $x \in X$, *we have*

$$\begin{array}{rcl} T_1 g(x) - T_1 h(x) & = & T_1(g - h)(x) \\ & = & r_1\{g(a_1 x) - h(a_1 x)\} \\ & \in & |r_1|.M.[V]_{seq} \end{array}$$

resp.

$$\begin{array}{rcl} T_2 g(x) - T_2 h(x) & = & T_2(g - h)(x) \\ & = & r_2\{g(a_2 x) - h(a_2 x)\} \\ & \in & |r_2|.M.[V]_{seq} \end{array}$$

So, for each $i = 1, 2$, *we have*

$$\delta(T_i g, T_i h) \leq |r_i|.\delta(g, h).$$

So, by hypothesis, the operator T_1 *(resp.* T_2*) is a contraction on* E_f, *so there exists a unique fixed points* g^1 *(resp* g^2*) of* T_1 *(resp.*T_2*), given for all* $x \in X$ *by*

$$g^1(x) = \lim_{n \longrightarrow +\infty} T_1^n f(x) \ (\ resp. g^2(x) = \lim_{n \longrightarrow +\infty} T_2^n f(x)\ ).$$

and

$$\delta(g^1, f) \leq \frac{|r_1|}{2 - 2|r_1|},$$

resp.

$$\delta(g^2, f) \leq \frac{1}{2-2|r_2|}.$$

For each $i = 1, 2$, *remplacing* (x, y) *by* $(a_i^n x, a_i^n y)$ *and multiplied by* r_i^n *in* (1) *with* $n \in \mathbb{N}$, *we have*

$$\Delta_{a,b,c} T_i^n f(x, y) \in |r_i|^n.V.$$

By taking the limits, our proof it's complete.

Theorem 4. *Let* X *be a linear space,* Y *be a sequentially complete Hausdorff topological vector space over* $\mathbb{R}$ *and* V *a nonempty bounded, convex and symetric set of* Y : $[[V]_{seq}]_{seq} = [V]_{seq}$.
Let $f : X \longrightarrow Y$ *be a application such that* $f(0) = 0$ *and satisfies* (2).
Suppose that there exists $L_1 > 0$ *(resp.* $L_2 > 0$*) such that, for all* $x, y \in X$

$$\varphi(a_1 x, a_1 y) \leq L_1 \varphi(x, y) \text{ (resp. } \varphi(a_2 x, a_2 y) \leq L_2 \varphi(x, y)).$$

and

$$L_1.|r_1| < 1 \text{ (resp. } L_2.|r_2| < 1 \text{)}.$$

Then, there exists a unique application $g^1 : X \longrightarrow Y$ *(resp.* $g^2 : X \longrightarrow Y$*) which is solution of* $(E-Q)(a, b, c)$ *such that* $g^1(0) = 0$ *(resp.* $g^2(0) = 0$*) and for all* $x \in X$, *we have*

$$g^1(x) - f(x) \in \frac{|r_1|}{2-2L_1.|r_1|}.\eta(x).[V]_{seq},$$

resp.

$$g^2(x) - f(x) \in \frac{L_2}{2-2L_2.|r_2|}.\eta(x).[V]_{seq}.$$

Proof By symmetry of V, *for all* $x \in X$, *we have*

$$r_1 f(a_1 x) - f(x) \in \frac{|r_1|}{2}.\eta(x).V,$$

resp.

$$r_2 f(a_2 x) - f(x) \in \frac{L_2}{2}.\eta(x).V.$$

Consider

$$E_f := \{g : X \longrightarrow Y;\ \exists M = M_g \in \mathbb{R}^+ :\ g(x) - f(x) \in M.\eta(x).[V]_{seq}, \quad \forall x \in X\}.$$

and we define the metric on E_f *by*

$$\delta(g, h) := \inf\{M \in \mathbb{R}^+;\ g(x) - h(x) \in M.\eta(x).[V]_{seq}\}, \quad \forall g, h \in E_f$$

We can show that (E_f, δ) is a complete metric space.
For each $i = 1, 2$, we define

$$T_i : E_f \longrightarrow E_f,\ g \longmapsto T_i g(x) = r_i g(a_i x), \quad \forall \in X$$

The operators T_1 (resp. T_2) is well defined : Let $g \in E_f$ and $x \in X$, using the convexity of $[V]_{seq}$, we have

$$\begin{array}{rcl} T_1 g(x) - f(x) & = & T_1(g - f)(x) + T_1 f(x) - f(x) \\ & \in & \{M' L_1 + \frac{1}{2}\}|r_1|.\eta(x).[V]_{seq} \end{array}$$

resp.

$$\begin{array}{rcl} T_2 g(x) - f(x) & = & T_2(g - f)(x) + T_2 f(x) - f(x) \\ & \in & \{M'|r_2| + \frac{1}{2}\}L_2.\eta(x).[V]_{seq} \end{array}$$

Now, let $(g, h) \in (E_f \times E_f)$, M' such that $\delta(g, h) < M'$, for all $x \in X$, we have

$$\begin{array}{rcl} T_1 g(x) - T_1 h(x) & = & T_1(g - h)(x) \\ & = & r_1\{g(a_1 x) - h(a_1 x)\} \\ & \in & L_1.|r_1|.M.\eta(x).[V]_{seq} \end{array}$$

resp.

$$\begin{array}{rcl} T_2 g(x) - T_2 h(x) & = & T_2(g - h)(x) \\ & = & r_2\{g(a_2 x) - h(a_2 x)\} \\ & \in & L_2.|r_2|.M.\eta(x).[V]_{seq} \end{array}$$

So, for each $i = 1, 2$, we have

$$\delta(T_i g, T_i h) \leq L_i |r_i|.\delta(g, h)$$

So, by hypothesis, the operator T_1 (resp. T_2) is a contraction on E_f, so there exists a unique fixed point g^1 of T_1 (resp. g^2 of T_2), given for all $x \in X$ by

$$g^1(x) = \lim_{n \longrightarrow +\infty} T_1^n f(x) \ (resp. g^2(x) = \lim_{n \longrightarrow +\infty} T_2^n f(x) \ ).$$

and

$$\delta(g^1, f) \leq \frac{|r_1|}{2 - 2L_1.|r_1|}$$

resp.

$$\delta(g^2, f) \leq \frac{L_2}{2 - 2L_2.|r_2|}$$

For each $i = 1, 2$*, remplacing in* (2)*,* (x, y) *by* $(a_i^n x, a_i^n y)$ *and multiplied by* r_i^n *with* $n \in \mathbb{N}$*, we have*

$$\Delta_{a,b,c} T_i^n f(x, y) \in (L_i |r_i|)^n \varphi(x, y) V, \quad \forall x, y \in X.$$

By passing the limits, our proof it's complete.
Claim (E_f, δ) *is a complete metric spece.*

Proof. It's clair that E_f *is nonempty and* δ *it's a metric on* E_f*.*
Let $(g_n)_{n\in\mathbb{N}}$ *be Cauchy sequence on* (E_f, δ)*.*
Let $x \in X$*,* U *be neighbourhood of* $0 \in Y$ *and* $\lambda > 0$ *such that* $\lambda.\eta(x).[V]_{seq} \subseteq U$*.*
Since $(g_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence, let* $N = N_\lambda$ *such that :* $\forall m \geq n \geq N \in \mathbb{N}$*, on a*

$$\delta(g_n, g_m) \leq \lambda.$$

Hence

$$g_m(x) - g_n(x) \in \lambda.\eta(x).[V]_{seq} \subseteq U, \quad \forall m \geq n \geq N.$$

So, for every $x \in X$ *the sequence* $(g_n(x))_{n\in\mathbb{N}}$ *is a Cauchy sequence* Y*, so converge.*
Let $g : X \longrightarrow Y$ *be the application defined by*

$$g(x) = \lim_{n\longrightarrow+\infty} g_n(x), \quad \forall x \in X.$$

Let $\alpha > 0$ *and* $N_\alpha \in \mathbb{N}$ *such that*

$$\delta(g_m, g_n) \leq \alpha, \quad \forall m \geq n \geq N_\alpha \in \mathbb{N}.$$

i.e

$$g_m(x) - g_n(x) \in \alpha.\eta(x).[V]_{seq}, \quad \forall x \in X,\ \forall m > n \geq N_\alpha.$$

Fix $n \in \mathbb{N}$ *and take* $m \to \infty$ *we have*

$$g(x) - g_n(x) \in \alpha.\eta(x).[[V]_{seq}]_{seq} = \alpha.\eta(x).[V]_{seq}, \quad \forall x \in X.$$

So

$$\lim_{n\longrightarrow+\infty} \delta(g_n, g) = 0.$$

On the other hand, for $\alpha = 1$ *there exists* $N \in \mathbb{N}$ *and* $M \in \mathbb{R}^+$ *such that, for all* $x \in X$*, we have*

$$g_N(x) - f(x) \in M.\eta(x).[V]_{seq} \text{ and } g_N(x) - g(x) \in \eta(x).[V]_{seq}.$$

Hence

$$g(x) - f(x) \in \{M+1\}.\eta(x).[V]_{seq}, \quad \forall x \in X.$$

i.e $g \in E_f$.

Remark Taking Y a real Banach space, then for $V = \overline{B}(0,\varepsilon)$, $\overline{B}(0,\varphi)$, all result's given recently by M.Kumar and A.Kumar are a simple applications of our's.

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Mohamed Akkouchi and Abdelkarim Ed-Darraz*
Department of Mathematics
University of Cadi Ayyad
Av. du prince My. Abdellah B.P 2390. Marrakech, Morocco.
Email:*akkouchimo@yahoo.fr and a.l.eddarraz@gmail.com*
* Corresponding author.