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JENSEN'S INEQUALITY FOR MIXED CONVEX FUNCTIONS OF TWO REAL VARIABLES

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Abstract. The aim of our paper is to extend the validity of Jensen's inequality outside the usual convexity, in the larger framework of mixed convex functions.

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1.Introduction

In recent years some authors have studied the possibility to extend Jensen's inequality to the framework of mixed convex functions. See [1], [2], [3]. The Jensen's inequality is in fact much more general, as shows the following simple remark of C. P. Niculescu and C. I. Spiridon:

Remark No.1 [4] Suppose that K is a convex subset of the Euclidean space $\mathbb{R}^N$ carrying a Borel probability measure μ. Then every μ-integrable function $f : K \to \mathbb{R}$ that admits a supporting hyperplane at the barycenter of μ,

$$b_\mu = \int_K x \mathrm{d}\mu(x),$$

verifies the Jensen inequality

$$f(b_\mu) \leq \int_K f(x)\mathrm{d}\mu(x).$$

It is well known that the convexity yields the existence of a supporting hyperplane at all interior points of the domain. We will establish several more general conditions such that the previous obsevation to take place in $\mathbb{R}^2$.

Considering the case of functions of one variable, the following generalization of Jensen's inequality for mixed convex functions of one real variable holds.

Theorem No.1 [4] *Let $f : [a,b] \longrightarrow \mathbb{R}$ and the point $c \in \left[a, \frac{a+b}{2}\right]$ such that*

i) $f(c-x) + f(c+x) = 2f(c)$, for all $c \pm x \in [a,b]$;

ii) $f_{/[c,b]}$ *is a convex function.*
Then for every Borel probability measure μ *on* $[a,b]$*, such that*

$$\int_a^b x \mathrm{d}\mu(x) + a \geq 2c \tag{1}$$

we have the inequality

$$f\left(\int_a^b x \mathrm{d}\mu(x)\right) \leq \int_a^b f(x) \mathrm{d}\mu(x). \tag{2}$$

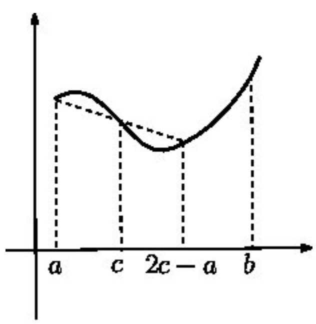
The last inequality works in the reverse way when $f_{/[c,b]}$ *is concave.*

We will call *mixed convex* all functions which satisfy this type of assumptions. In order to make this paper self-contained, we reproduce here the proof of this result.

Proof. The case where $c = a$ is covered by classical inequality of Jensen so we may assume that $c \in \left(a, \frac{a+b}{2}\right)$. In this case the point $2c - a$ is in interior to $[a,b]$. By hypothesis, the barycenter b_μ lies in the interval $[2c-a, b]$. If $b_\mu = b$, then $\mu = \delta_b$ and the conclusion is clear. If b_μ is interior to $[a,b]$, it is denote by h the affine function joining the points $(a, f(a))$ and $(2c-a, f(2c-a))$.
We define the function $g : [a,b] \longrightarrow \mathbb{R}$

$$g(x) = \begin{cases} h(x), & \text{if } x \in [a, 2c-a] \\ f(x), & \text{if } x \in [2c-a, b] \end{cases} .$$

See Figure 1.



Clearly, g is convex and this fact motivates the existence of a support line l of g at b_μ. Since $h \geq f$, then l is a support line at b_μ also for f. By the remark above, this ends the proof.

Our goal is to extend this result for mixed convex functions of two real variables. Before stating the result we establish the notation and recall some definitions from the literature. We consider the following two-dimensional interval $\Delta = [-a, b] \times [c, d]$ of $\mathbb{R}^2$, with $0 < a < b$ and $c < d$. The function $f : \Delta \longrightarrow \mathbb{R}$ is convex in the usual sense if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for every $x, y \in \Delta$ and $t \in [0, 1]$. The function f is concave, if $-f$ is convex.

The function $f : \Delta \longrightarrow \mathbb{R}$ is co-ordinated convex on Δ if the functions

$$f_y : [-a, b] \longrightarrow \mathbb{R},\ f_y(u) = f(u, y)$$

and

$$f_x : [c, d] \longrightarrow \mathbb{R},\ f_x(v) = f(x, v)$$

are convex for every $x \in [-a, b]$, respectively every $y \in [c, d]$.

2.Main result

The function $f : [-a, a] \times [c, d] \longrightarrow \mathbb{R}$ is symmetric with respect to the axis Oy if

$$f(-x, y) = -f(x, y)$$

for all $(x, y) \in [-a, a] \times [c, d]$.

We are in position to prove the following generalization of Theorem for mixed convex functions of two real variables.

Theorem No.2 *Let* $\Delta = [-a, b] \times [c, d]$*, with* $0 \leq a < b,\ c < d$*, endowed with a Borel probability measure* μ *with* $\mathrm{d}\mu(x, y) = p(x)q(y)\mathrm{d}y\mathrm{d}x$*, such that* $p(x)q(y) > 0$*,* $\int_{-a}^{b} \int_{c}^{d} p(x)q(y)\mathrm{d}y\mathrm{d}x = 1$ *and its barycenter*

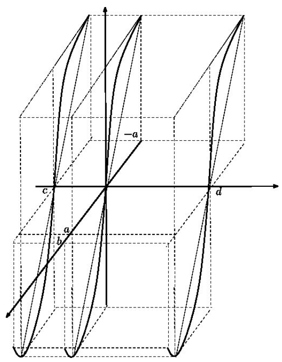
$$b_\mu \left(\int_{-a}^{b} \int_{c}^{d} xp(x)q(y)\mathrm{d}y\mathrm{d}x, \int_{-a}^{b} \int_{c}^{d} yp(x)q(y)\mathrm{d}y\mathrm{d}x \right) \in [a, b] \times [c, d].$$

Then for any function f *which is* μ*-integrable, symmetric with respect to the axis* Oy *on* $[-a, a] \times [c, d]$ *and convex on* $[0, b] \times [c, d]$*, it holds*

$$f(b_\mu) \leq \int_{-a}^{b} \int_{c}^{d} f(x, y)p(x)q(y)\mathrm{d}y\mathrm{d}x.$$

Proof. We apply the same recipe as in the proof of Theorem . If $a = 0$, this is a consequence of the classic Jensen's inequality, applied in the two-dimensional Euclidean space.

Suppose that $a \neq 0$, which implies that $a \in (-a, b)$.



According to the hypothesis, the barycenter b_μ belongs to $[a,b]\times[c,d]$ and we consider the affine function h determined by the points $M\,(a, d, f(a,d))$, $M'(-a, d, f(-a,d))$, respectively $O(0,0,0)$,

$$h : [-a, a] \times [c, d] \longrightarrow \mathbb{R}, h(x,y) = \frac{f(a,d)}{a}x.$$

See Figure 2. Let $g : \Delta \longrightarrow \mathbb{R}$ be defined by

$$g\,(x) = \begin{cases} h\,(x,y)\,, & \text{if } (x,y) \in [-a,a] \times [c,d] \\ f\,(x,y)\,, & \text{if } (x,y) \in [a,b] \times [c,d] \end{cases}.$$

Obviously, the function g is convex and therefore admits a supporting hyperplane at the point b_μ such that $f(b_\mu) = h(b_\mu)$ and $f(x,y) \geq h(x,y)$, for all $(x,y) \in \Delta$. Then

$$\begin{aligned} f(b_\mu) = h(b_\mu) = h\left(\int_{-a}^{b}\int_{c}^{d} xp(x)q(y)\mathrm{d}y\mathrm{d}x, \int_{-a}^{b}\int_{c}^{d} yp(x)q(y)\mathrm{d}y\mathrm{d}x\right) \\ = \int_{-a}^{b}\int_{c}^{d} h(x,y)p(x)q(y)\mathrm{d}y\mathrm{d}x \leq \int_{-a}^{b}\int_{c}^{d} f(x,y)p(x)q(y)\mathrm{d}y\mathrm{d}x \end{aligned}$$

and the proof is completed.

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