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THE HADAMARD PRODUCT IN GENERALIZED $U^N(P,Q)$-MATRICES

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Abstract. In this paper we studied the generalized Fibonacci and Lucas matrix $U^n(p,q)$, and we defined $U^n(p,q) \circ U^{-n}(p,q)$, Hadamard product of $U^n(p,q)$ matrix and $U^{-n}(p,q)$ matrix. We investigated some properties of Hadamard product of generalized Fibonacci and Lucas matrices.

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1. Introduction

In Horadam notation [3], we consider a sequence $\{W_n(a,b,p,q)\}$, or briefly $\{W_n\}$, defined by the recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, n \geq 2, \tag{1}$$

with $W_0 = a$, $W_1 = b$, where a, b, p and q are integers with $p > 0$, $q \neq 0$, and $\Delta = p^2 - 4q > 0$. We are interested in the following two special cases of $\{W_n\}$: $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$, and $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = p$. It is well known that $\{U_n\}$ and $\{V_n\}$ can be expressed in the form

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, V_n = \alpha^n + \beta^n, \tag{2}$$

where $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$.

Especially, if $p = -q = 1$ and $2p = -q = 2$, $\{U_n\}$ is the usual Fibonacci and Jacobsthal sequence, respectively.

We define $U(p,q)$ be the 2×2 matrix

$$U(p,q) = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}, \tag{3}$$

then for an integer n with $n \geq 1$, $U^n(p,q)$ has the form

$$U^n(p,q) = \begin{bmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{bmatrix}. \tag{4}$$

This property provides an alternate proof of Cassini Fibonacci formula:

$$U_{n+1}U_{n-1} - U_n^2 = -q^{n-1}.$$

Also, let n and m be two integers such that $m, n \geq 1$. The following results are obtained from the identity $U^{n+m}(p,q) = U^n(p,q)U^m(p,q)$ for the matrix (4):

$$U_{n+m+1} = U_{n+1}U_{m+1} - qU_nU_m, \tag{5}$$

$$U_{n+m} = U_nU_{m+1} - qU_{n-1}U_m. \tag{6}$$

In [1], the author define the Lucas $V(p,q)$-matrix by

$$V(p,q) = \begin{bmatrix} p^2 - 2q & -pq \\ p & -2q \end{bmatrix}. \tag{7}$$

It is easy to see that

$$\begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} = V(p,q) \begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} \text{ and } \Delta \begin{bmatrix} U_{n+1} \\ U_n \end{bmatrix} = V(p,q) \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix}$$

where U_n and V_n are as above. Our aim, is not to compute powers of matrices. Our aim is to find different relations between matrices containing generalized Fibonacci and Lucas numbers.

That is, we obtain relations between the generalized Fibonacci $U(p,q)$-matrix and the Lucas $V(p,q)$. In particular,

Theorem 1.1. *Let $V(p,q)$ be a matrix as in (7). Then, for all integers $n \geq 1$, the following matrix power is held below*

$$V^n(p,q) = \begin{cases} \Delta^{\frac{n}{2}} \begin{bmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{bmatrix} & if \;\; n \text{ even} \\ \Delta^{\frac{n-1}{2}} \begin{bmatrix} V_{n+1} & -qV_n \\ V_n & -qV_{n-1} \end{bmatrix} & if \;\; n \text{ odd,} \end{cases} \tag{8}$$

with $\Delta = p^2 - 4q$ and where U_n and V_n are the n-th generalized Fibonacci and Lucas numbers, respectively.

Proof. We use mathematical induction on n. First, we consider odd n. For $n = 1$,

$$V^1(p,q) = \begin{bmatrix} V_2 & -qV_1 \\ V_1 & -qV_0 \end{bmatrix},$$

since $V_2 = p^2 - 2q$, $V_1 = p$ and $V_0 = 2$. So, (8) is indeed true for $n = 1$. Now we suppose it is true for $n = k$, that is

$$V^k(p,q) = \Delta^{\frac{k-1}{2}} \begin{bmatrix} V_{k+1} & -qV_k \\ V_k & -qV_{k-1} \end{bmatrix}.$$

Using properties of the generalized Lucas numbers and the induction hypothesis, we can write

$$V^{k+2}(p,q) = V^k(p,q)V^2(p,q) = \Delta^{\frac{k+1}{2}} \begin{bmatrix} V_{k+3} & -qV_{k+2} \\ V_{k+2} & -qV_{k+1} \end{bmatrix},$$

as desired. Secondly, let us consider even n. For $n = 2$ we can write

$$V^2(p,q) = \Delta \begin{bmatrix} U_3 & -qU_2 \\ U_2 & -qU_1 \end{bmatrix}.$$

So, (8) is true for $n = 2$. Now, we suppose it is true for $n = k$, using properties of the generalized Fibonacci numbers and the induction hypothesis, we can write

$$V^{k+2}(p,q) = V^k(p,q)V^2(p,q) = \Delta^{\frac{k+2}{2}} \begin{bmatrix} U_{k+3} & -qU_{k+2} \\ U_{k+2} & -qU_{k+1} \end{bmatrix},$$

as desired. Hence, (8) holds for all n.

In this paper we studied the generalized Fibonacci and Lucas matrix $U^n(p,q)$, and we defined $U^n(p,q) \circ U^{-n}(p,q)$, Hadamard product of $U^n(p,q)$ matrix and $U^{-n}(p,q)$ matrix. We investigated some properties of Hadamard product of generalized Fibonacci and Lucas matrices.

2. Some properties of the $U^n(p,q) \circ U^{-n}(p,q)$ matrix

Let $U^n(p,q)$ be generalized Fibonacci matrix (4), and $U^{-n}(p,q)$ the inverse of $U^n(p,q)$. Then, the Hadamard product of $U^n(p,q)$ and $U^{-n}(p,q)$, denoted $U^n(p,q) \circ U^{-n}(p,q)$, is defined by

$$U^n(p,q) \circ U^{-n}(p,q) = q^{-n} U^n(p,q) \circ Adj(U^n(p,q)) \tag{9}$$

where $Adj(U^n(p,q))$ is the adjugate of the $U^n(p,q)$ matrix, and $\circ$ is the Hadamard product.

Definition 2.1. [5] *Let $A = (a_{ij})$ be $n \times n$ matrix over any commutative ring. The permanent of A, denoted by* per(A)*, is defined by*

$$\operatorname{per}(A) = \textstyle\sum_\sigma a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n},$$

where the summation extends over all one-to one functions from $\{1,2,..,n\}$ *to itself.*

Theorem 2.2. $\det[U^n(p,q)\circ U^{-n}(p,q)] = 1-2q^{-n+1}U_n^2$.
Proof. For all integer n, $U^n(p,q)\circ U^{-n}(p,q)$ is defined by

$$\begin{aligned} U^n(p,q)\circ U^{-n}(p,q) &= q^{-n}\begin{bmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{bmatrix}\begin{bmatrix} -qU_{n-1} & qU_n \\ -U_n & U_{n+1} \end{bmatrix} \\ &= -q^{-n}\begin{bmatrix} qU_{n+1}U_{n-1} & q^2U_n^2 \\ U_n^2 & qU_{n+1}U_{n-1} \end{bmatrix}, \end{aligned}$$

where U_n is the nth generalized Fibonacci numbers. Then,

$$\begin{aligned} \det[U^n(p,q)\circ U^{-n}(p,q)] &= q^{-2n}q^2(U_{n+1}U_{n-1}-U_n^2)(U_{n+1}U_{n-1}+U_n^2) \\ &= -q^{-2n+1}(U_{n+1}U_{n-1}-U_n^2)(-q(U_{n+1}U_{n-1}+U_n^2)) \\ &= -q^{-2n+1}(U_{n+1}U_{n-1}-U_n^2)\mathrm{per}(U^n(p,q)) \\ &= q^{-n}\mathrm{per}(U^n(p,q)), \end{aligned}$$

with $\mathrm{per}(U^n(p,q)) = -q(U_{n+1}U_{n-1}+U_n^2) = q^n - 2qU_n^2$. This completes the proof.

Corollary 2.3. $\mathrm{tr}[U^n(p,q)\circ U^{-n}(p,q)] = 2(1-q^{-n+1}U_n^2)$.
Proof. By considering the previous proof,

$$U^n(p,q)\circ U^{-n}(p,q) = -q^{-n}\begin{bmatrix} qU_{n+1}U_{n-1} & q^2U_n^2 \\ U_n^2 & qU_{n+1}U_{n-1} \end{bmatrix}. \tag{10}$$

Then, $\mathrm{tr}[U^n(p,q)\circ U^{-n}(p,q)] = -2q^{-n+1}U_{n+1}U_{n-1}$. Furthermore, by Cassini formula $U_{n+1}U_{n-1} = -q^{n-1}+U_n^2$, and we write

$$\mathrm{tr}[U^n(p,q)\circ U^{-n}(p,q)] = 2(1-q^{-n+1}U_n^2).$$

Theorem 2.4. *The eigenvalues of the matrix* $U^n(p,q)\circ U^{-n}(p,q)$ *are*

$$\lambda_1 = 1, \lambda_2 = q^{-n}\mathrm{per}(U^n(p,q)). \tag{11}$$

Proof. The characteristic polynomial of the matrix $U^n(p,q)\circ U^{-n}(p,q)$ is

$$\begin{aligned} \Lambda_{U^n(p,q)\circ U^{-n}(p,q)}(\lambda) &= \det\left(\lambda I - (U^n(p,q)\circ U^{-n}(p,q))\right) \\ &= \det\begin{bmatrix} \lambda + q^{-n+1}U_{n+1}U_{n-1} & q^{-n+2}U_n^2 \\ q^{-n}U_n^2 & \lambda+q^{-n+1}U_{n+1}U_{n-1} \end{bmatrix} \\ &= (\lambda+q^{-n+1}U_{n+1}U_{n-1})^2 - q^{-2n+2}U_n^4, \end{aligned}$$

where U_n is the nth generalized Fibonacci numbers. Hence,

$$\begin{aligned}\Lambda_{U^n(p,q)\circ U^{-n}(p,q)}(\lambda)=0 &\Leftrightarrow (\lambda+q^{-n+1}U_{n+1}U_{n-1})^2-q^{-2n+2}U_n^4=0\\ &\Leftrightarrow (\lambda-q^{-n}\mathrm{per}(U^n(p,q)))(\lambda-1)=0\end{aligned}$$

and the eigenvalues of $U^n(p,q)\circ U^{-n}(p,q)$ are $\lambda_1=1$, $\lambda_2=q^{-n}\mathrm{per}(U^n(p,q))$.

3. Eigenvalues of the matrix $U^n(p,q)\circ U^{-n}(p,q)$

If λ_i, $i=1,2$, an eigenvalue of the matrix $U^n(p,q)\circ U^{-n}(p,q)$, the corresponding eigenvectors v_i are the solutions of

$$\left(\lambda_i I-(U^n(p,q)\circ U^{-n}(p,q))\right)v_i=0. \tag{12}$$

We first calculate the eigenvector corresponding to $\lambda_1=1$. Then,

$$\begin{aligned}I-(U^n(p,q)\circ U^{-n}(p,q)) &= \begin{bmatrix} 1+q^{-n+1}U_{n+1}U_{n-1} & q^{-n+2}U_n^2\\ q^{-n}U_n^2 & 1+q^{-n+1}U_{n+1}U_{n-1}\end{bmatrix}\\ &= \begin{bmatrix} q^{-n+1}U_n^2 & q^{-n+2}U_n^2\\ q^{-n}U_n^2 & q^{-n+1}U_n^2\end{bmatrix}.\end{aligned}$$

From (12),

$$\begin{bmatrix} q^{-n+1}U_n^2 & q^{-n+2}U_n^2\\ q^{-n}U_n^2 & q^{-n+1}U_n^2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}.$$

By using elementary row operations, the coefficients matrix of this homogeneous system becomes

$$\begin{bmatrix} 1 & q\\ 0 & 0\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}.$$

Since the rank of the coefficients matrix of this homogeneous system is equal to 1, there exist infinitely many solutions dependent on one parameter. The solution to this set of equations is $x=-qy=-qt$, where t is arbitrary. In this case, linearly independent eigenvector corresponding to $\lambda_1=1$ is equal to $v_1=(-q,1)^t$.

Now calculate the eigenvector to $\lambda_2=q^{-n}\mathrm{per}(U^n(p,q))$. Then,

$$\begin{bmatrix} -q^{-n+1}U_n^2 & q^{-n+2}U_n^2\\ q^{-n}U_n^2 & -q^{-n+1}U_n^2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}.$$

By using elementary row operations, the coefficients matrix of this homogeneous system becomes

$$\begin{bmatrix} 1 & -q\\ 0 & 0\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}=\begin{bmatrix} 0\\ 0\end{bmatrix}.$$

The solution to this set of equations is $x = qy = qt$, where t is arbitrary. In this case, linearly independent eigenvector corresponding to $\lambda_2 = q^{-n}\mathrm{per}(U^n(p,q))$ is equal to $v_2 = (q,1)^t$.

Observation 3.1. *The matrix $U^n(p,q) \circ U^{-n}(p,q)$ is diagonalizable. In view of the above, we can write*

$$P = \begin{bmatrix} -q & q \\ 1 & 1 \end{bmatrix},$$

to obtain

$$P^{-1}(U^n(p,q) \circ U^{-n}(p,q))P = \begin{bmatrix} 1 & 0 \\ 0 & q^{-n}\mathrm{per}(U^n(p,q)) \end{bmatrix}. \tag{13}$$

Let M_n denote the class of complex $n \times n$ matrices.
Definition 3.2. [4] *The l_1 norm on M_n is defined by*

$$||A||_1 = \sum_{i,j=1}^{n} |a_{ij}|,$$

and the Euclidean norm or l_2 norm is

$$||A||_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Example 3.3. For each integer n, the Fibonacci matrix $U^n(1,-1)$ satisfies

1. $||U^n(1,-1) \circ U^{-n}(1,-1)||_1 = 2((-1)^n + 2F_n^2)$
2. $||U^n(1,-1) \circ U^{-n}(1,-1)||_2 = (4F_n^4 + (-1)^n 4F_n^2 + 2)^{\frac{1}{2}}$

where F_n is the nth Fibonacci numbers.

Theorem 3.4. *The matrix $U^n(p,q) \circ U^{-n}(p,q)$ is invertible and*

$$\left[U^n(p,q) \circ U^{-n}(p,q)\right]^{-1} = \begin{bmatrix} \frac{1-q^{-n+1}U_n^2}{1-2q^{-n+1}U_n^2} & \frac{q^{-n+2}U_n^2}{1-2q^{-n+1}U_n^2} \\ \frac{q^{-n}U_n^2}{1-2q^{-n+1}U_n^2} & \frac{1-q^{-n+1}U_n^2}{1-2q^{-n+1}U_n^2} \end{bmatrix}. \tag{14}$$

Proof. For each integer n, the adjugate of $U^n(p,q) \circ U^{-n}(p,q)$ is

$$Adj(U^n(p,q) \circ U^{-n}(p,q)) = -q^{-n} \begin{bmatrix} qU_{n+1}U_{n-1} & -q^2U_n^2 \\ -U_n^2 & qU_{n+1}U_{n-1} \end{bmatrix}.$$

By using Cassini formula, we obtain $U_{n+1}U_{n-1} = -q^{n-1} + U_n^2$. For

$$\begin{bmatrix} 1 - q^{-n+1}U_n^2 & q^{-n+2}U_n^2 \\ q^{-n}U_n^2 & 1 - q^{-n+1}U_n^2 \end{bmatrix} \tag{15}$$

and from Theorem (2.2),

$$\det(U^n(p,q) \circ U^{-n}(p,q)) = q^{-n}\mathrm{per}(U^n(p,q)) = 1 - 2q^{-n+1}U_n^2.$$

Then,

$$\left[U^n(p,q) \circ U^{-n}(p,q)\right]^{-1} = \frac{1}{1 - 2q^{-n+1}U_n^2} \begin{bmatrix} 1 - q^{-n+1}U_n^2 & q^{-n+2}U_n^2 \\ q^{-n}U_n^2 & 1 - q^{-n+1}U_n^2 \end{bmatrix}.$$

4. Special case of the $R(p,q)$ matrix

In general, by induction there is a way to build arrays of type $U^n(p,q)$.

Theorem 4.1. *If A is a square matrix with $A^2 = pA - qI$ and I matrix identity of order 2. Then, $A^n = U_nA - qU_{n-1}I$, for all $n \in \mathbb{Z}$.*

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $A^n = U_nA - qU_{n-1}I$, for every n. We now show that $A^{-n} = U_{-n}A - qU_{-n-1}I$ for every $n \in \mathbb{N}$. Let $B = pI - A = qA^{-1}$, then

$$B^2 = (pI - A)^2 = p^2I - 2pA + A^2 = p(pI - A) - qI = pB - qI,$$

this shows that $B^n = U_nB - qU_{n-1}I$. That is, $(qA^{-1})^n = U_n(pI - A) - qU_{n-1}I$. Therefore $q^nA^{-n} = -U_nA + (pU_n - qU_{n-1})I = -U_nA + U_{n+1}I$. Thus,

$$A^{-n} = -q^{-n}U_nA + q^{-n}U_{n+1}I = U_{-n}A - qU_{-n-1}I.$$

Thus, the proof is completed.

The well-known identity

$$U_{n+1}^2 - qU_n^2 = U_{2n+1} \tag{16}$$

has as its Lucas counterpart

$$V_{n+1}^2 - qV_n^2 = \Delta U_{2n+1}. \tag{17}$$

Indeed, since $V_{n+1} = U_{n+2} - qU_n = pU_{n+1} - 2qU_n$ and $V_n = 2U_{n+1} - pU_n$, the equation (17) follows from (16). We define $R(p,q)$ be the 2×2 matrix

$$R(p,q) = \frac{1}{2}\begin{bmatrix} p & \Delta \\ 1 & p \end{bmatrix}, \tag{18}$$

then for an integer n, $R^n(p,q)$ has the form

$$R^n(p,q) = \frac{1}{2}\begin{bmatrix} V_n & \Delta U_n \\ U_n & V_n \end{bmatrix}. \tag{19}$$

Theorem 4.2. *$V_n^2 - \Delta U_n^2 = 4q^n$, for all $n \in \mathbb{Z}$.*

Proof. Since $\det(R(p,q)) = q$, $\det(R^n(p,q)) = (\det(R(p,q)))^n = q^n$. Moreover, since (19), we get $\det(R^n(p,q)) = \frac{1}{4}(V_n^2 - \Delta U_n^2)$. The proof is completed.

Let us give a different proof of one of the fundamental identities of Generalized Fibonacci and Lucas numbers, by using the matrix $R^n(p,q)$ and $R^{-n}(p,q)$. Then, the Hadamard product $R^n(p,q) \circ R^{-n}(p,q)$, satisfies

Theorem 4.3. $\det\left[R^n(p,q) \circ R^{-n}(p,q)\right] = 1 + \frac{\Delta U_n^2}{2q^n}$.

Proof. For all integer n, $R^n(p,q) \circ R^{-n}(p,q)$ is defined by

$$\begin{aligned} R^n(p,q) \circ R^{-n}(p,q) &= \frac{1}{4q^n}\begin{bmatrix} V_n & \Delta U_n \\ U_n & V_n \end{bmatrix}\begin{bmatrix} V_n & -\Delta U_n \\ -U_n & V_n \end{bmatrix} \\ &= \frac{1}{4q^n}\begin{bmatrix} V_n^2 & -(\Delta U_n)^2 \\ -U_n^2 & V_n^2 \end{bmatrix}, \end{aligned}$$

where $\Delta = p^2 - 4q$, and U_n, V_n are the nth generalized Fibonacci and Lucas numbers, respectively. Then,

$$\begin{aligned} \det[R^n(p,q) \circ R^{-n}(p,q)] &= \frac{1}{(4q^n)^2}(V_n^2 - \Delta U_n^2)(V_n^2 + \Delta U_n^2) \\ &= \frac{1}{8q^{2n}}(V_n^2 - \Delta U_n^2)\frac{(V_n^2 + \Delta U_n^2)}{2} \\ &= \frac{1}{8q^{2n}}(V_n^2 - \Delta U_n^2)\mathrm{per}(R^n(p,q)) \\ &= \frac{1}{2q^n}\mathrm{per}(R^n(p,q)), \end{aligned}$$

with $\mathrm{per}(R^n(p,q)) = \frac{V_n^2 + \Delta U_n^2}{2} = 2q^n + \Delta U_n^2$. This completes the proof.

Corollary 4.4. $\mathrm{tr}[R^n(p,q) \circ R^{-n}(p,q)] = 2 + \frac{\Delta U_n^2}{2q^n}$.

Proof. It is easy to see that $\mathrm{tr}[R^n(p,q) \circ R^{-n}(p,q)] = \frac{1}{2q^n}V_n^2$. Furthermore, by theorem (4.2), $V_n^2 = 4q^n + \Delta U_n^2$, and we write

$$\mathrm{tr}[R^n(p,q) \circ R^{-n}(p,q)] = 2 + \frac{\Delta U_n^2}{2q^n}.$$

Theorem 4.5. *The eigenvalues of the matrix* $R^n(p,q)\circ R^{-n}(p,q)$ *are*

$$\lambda_1 = 1, \lambda_2 = \frac{\mathrm{per}(R^n(p,q))}{2q^n}. \tag{20}$$

Proof. The characteristic polynomial of the matrix $R^n(p,q)\circ R^{-n}(p,q)$ is

$$\begin{aligned}\Lambda_{R^n(p,q)\circ R^{-n}(p,q)}(\lambda) &= \det\left(\lambda I - (R^n(p,q)\circ R^{-n}(p,q))\right)\\ &= \det\begin{bmatrix} \lambda - \frac{V_n^2}{4q^n} & \frac{(\Delta U_n)^2}{4q^n} \\ \frac{U_n^2}{4q^n} & \lambda - \frac{V_n^2}{4q^n}\end{bmatrix}\\ &= \left(\lambda - \frac{V_n^2}{4q^n}\right)^2 - \frac{\Delta^2 U_n^4}{(4q^n)^2},\end{aligned}$$

where V_n is the nth generalized Lucas numbers. Hence,

$$\begin{aligned}\Lambda_{U^n(p,q)\circ U^{-n}(p,q)}(\lambda) = 0 &\Leftrightarrow \lambda^2 - \frac{V_n^2}{2q^n}\lambda + \frac{\mathrm{per}(R^n(p,q))}{2q^n} = 0\\ &\Leftrightarrow \left(\lambda - \frac{\mathrm{per}(R^n(p,q))}{2q^n}\right)(\lambda - 1) = 0\end{aligned}$$

and the eigenvalues of $R^n(p,q)\circ R^{-n}(p,q)$ are $\lambda_1 = 1$, $\lambda_2 = \frac{\mathrm{per}(R^n(p,q))}{2q^n}$.

Theorem 4.6. *The matrix* $R^n(p,q)\circ R^{-n}(p,q)$ *is invertible and*

$$\left[R^n(p,q)\circ R^{-n}(p,q)\right]^{-1} = \begin{bmatrix} \frac{V_n^2}{4q^n+2\Delta U_n^2} & \frac{(\Delta U_n)^2}{4q^n+2\Delta U_n^2} \\ \frac{U_n^2}{4q^n+2\Delta U_n^2} & \frac{V_n^2}{4q^n+2\Delta U_n^2}\end{bmatrix}. \tag{21}$$

Proof. For each integer n, the adjugate of $R^n(p,q)\circ R^{-n}(p,q)$ is

$$Adj(R^n(p,q)\circ R^{-n}(p,q)) = \frac{1}{4q^n}\begin{bmatrix} V_n^2 & (\Delta U_n)^2 \\ U_n^2 & V_n^2\end{bmatrix},$$

and from Theorem (4.3), $\det(R^n(p,q)\circ R^{-n}(p,q)) = 1 + \frac{\Delta U_n^2}{2q^n}$. Then,

$$\left[R^n(p,q)\circ R^{-n}(p,q)\right]^{-1} = \frac{1}{4q^n + 2\Delta U_n^2}\begin{bmatrix} V_n^2 & (\Delta U_n)^2 \\ U_n^2 & V_n^2\end{bmatrix}.$$

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