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REGULARIZATION AND HÖLDER TYPE ERROR ESTIMATES FOR AN INITIAL INVERSE HEAT PROBLEM WITH TIME-DEPENDENT COEFFICIENT

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Abstract. This paper discusses the initial inverse heat problem (backward heat problem) with time-dependent coefficient. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. Two regularization solutions of the backward heat problem will be given by a modified quasi-boundary value method. The Hölder type error estimates between the regularization solutions and the exact solution are obtained.

Keywords and phrases: Backward heat problem, Ill-posed problem, Nonhomogeneous heat, Contraction principle.

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1. Introduction

In this paper, we consider the problem of finding the temperature $u(x,t)$, $(x,t) \in (0,\pi) \times [0,T]$, such that

$$\begin{cases} \frac{\partial u}{\partial t} = b(t)\frac{\partial^2 u}{\partial x^2}, (x,t) \in (0,\pi) \times (0,T) \\ u(0,t) = u(\pi,t) = 0, t \in (0,T) \\ u(x,T) = g(x), x \in (0,\pi) \end{cases} \tag{1}$$

where $b(t), g(x)$ are given. The problem is called the backward heat problem with time-dependent coefficient. It is well known that this is an ill-posed problem. The goal is to set up a regularization process that makes this problem well posed in the sense of Hadamard:
(1) There is a solution.
(2) The solution is unique.
(3) The solution depends continuously on the data.

In the simple case $b(t) = 1$, the problem (1) is investigated in many papers, such as Clark and Oppenheimer [3], Denche and Bessila [4], Tautenhahn et al [21] Melnikova et al [11, 12], Trong et al [17, 18], B.Yildiz et al [22, 23].

Although there are many papers on the backward heat equation with the constant coefficient, but there are rarely works considered the backward heat with the time-dependent coefficient, such as (1). A few works of analytical methods were presented for this problem, for example [14]. However, the authors in [14] only used numerical computation method and the stability theory with explicitly error estimate has not been generalized accordingly. In the present paper, we want to determine the temperature $u(x,t)$ for $0 \le t < T$ by a modified quasi-boundary value method with two other approximation problems. Both methods of proving stability estimates are constructive: we construct stable solutions to the problem that can be numerically implemented. However, we do not pursue this aspect in this paper, as our aim here is to obtain stability estimates only. The numerical computation will be considered in our future research.

This paper is organized as follows: In Section 2, we simply analyze the ill-posedness of the problem (1) and conditions (1), (2) of Hadamard are addressed in this section. In Sections 3 and 4, we introduce two regularization solutions and establish some error estimates between the exact solution and the regularization solutions, respectively.

2.The ill-posedness of the backward heat problem

We suppose that $b(t) : [0,T] \to R$ is a continuous function on $[0,T]$ satisfying $0 < B_1 \le b(t) \le B_2, \forall t \in [0,T]$. Throughout this article, we denote by $\|.\|$ the L^2-norm and $<,>$ denote inner product on $L^2(0,\pi)$. We also suppose that $f \in L^2((0,T); L^2(0,\pi))$ and $g \in L^2(0,\pi)$.

Let $0 = q < \infty$. By $H^q(0,\pi)$ we denote the space of all functions $g \in L^2(0,\pi)$ with the property

$$\sum_{p=1}^{\infty}(1+p^2)^q|g_p|^2 < \infty,$$

where $g_p = \frac{2}{\pi}\int_0^\pi g(x)\sin(px)dx$. Then we also define $\|g\|^2_{H^q(0,\pi)} = \sum_{p=1}^{\infty}(1+p^2)^q|g_p|^2$. If $q = 0$ then $H^q(0,\pi)$ is $L^2(0,\pi)$. In the following Theorem, we consider the existence condition of solution to the problem (1).

Theorem 2.1. *The problem (1) has a unique solution u if and only if*

$$\sum_{p=1}^{\infty}\exp\left(2p^2\int_0^T b(s)ds\right)g_p^2 < \infty. \quad (2)$$

Proof. Suppose the Problem (1) has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, then u can be formulated in the frequency domain

$$u(x,t) = \sum_{p=1}^{\infty} \exp\left(p^2 \int_t^T b(\xi)d\xi\right) g_p^2 \sin(px). \tag{3}$$

This implies that

$$u_p(t) =< u(x,t), \frac{2}{\pi}\sin px >= \exp\left(p^2 \int_t^T b(\xi)d\xi\right) g_p.$$

Therefore

$$u_p(0) = \exp\left(p^2 \int_0^T b(s)ds\right) g_p. \tag{4}$$

Then

$$\|u(.,0)\|^2 = \frac{\pi}{2}\sum_{p=1}^{\infty} \exp\left(2p^2 \int_0^T b(s)ds\right) g_p^2 < \infty.$$

If (2) holds, then define $v(x)$ be as the function

$$v(x) = \sum_{p=1}^{\infty} \exp\left(p^2 \int_0^T b(s)ds\right) g_p \sin(px).$$

It is easy to see that $v \in L^2(0,\pi)$. Then, we consider the problem of finding u from the original value v

$$\begin{cases} u_t - b(t)u_{xx} = 0, \\ u(0,t) = u(\pi,t) = 0, \ t \in (0,T) \\ u(x,0) = v(x), \ x \in (0,\pi). \end{cases} \tag{5}$$

The problem (5) is the direct problem so it has a unique solution u (See [5]). We have

$$u(x,t) = \sum_{p=1}^{\infty} \left(\exp\{-p^2 \int_0^t b(\xi)d\xi\} < v(x), \sin px >\right) \sin px.$$

Thus

$$u(x,T) = \sum_{p=1}^{\infty} \left(\exp\{-p^2 \int_0^T b(\xi)d\xi\} < v(x), \sin px >\right) \sin px. \tag{6}$$

Therefore, we get

$$< v(x), \sin px >= \exp\left(p^2 \int_0^T b(s)ds\right) g_p. \quad (7)$$

Since (6), (7) and by a simple computation, we get

$$u(x,T) \;\; = \;\; \sum_{p=1}^{\infty} g_p \sin px = g(x).$$

Hence, u is the unique solution of (1).

Theorem 2.2 *The Problem (1) has at most one solution in* $C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$. *If (1) has a solution* u *then* u *is defined by*

$$u(x,t) = \sum_{p=1}^{\infty} \exp\left(p^2 \int_t^T b(\xi)d\xi\right) g_p \sin(px). \quad (8)$$

Proof.
The proof is divide into two step.
Step 1. The Problem (1) has at most one solution.
Let $u(x,t)$, $v(x,t)$ be two solutions of Problem (1) such that $u, v \in C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$. Put $w(x,t) = u(x,t) - v(x,t)$. Then w satisfies the equation

$$\begin{cases} w_t - b(t)w_{xx} = 0, \\ w(0,t) = w(\pi,t) = 0, \; t \in (0,T) \\ w(x,0) = 0, \; x \in (0,\pi). \end{cases} \quad (9)$$

Now, setting $G(t) = \int_0^\pi w^2(x,t)dx \quad (0 \le t \le T)$, and by taking the derivative of $G(t)$, we get

$$G'(t) = 2\int_0^\pi w(x,t)w_t(x,t)dx = 2b(t)\int_0^\pi w(x,t)w_{xx}(x,t)dx.$$

Using Green formula, we obtain

$$G'(t) = -2b(t)\int_0^\pi w_x^2(x,t)dx. \quad (10)$$

By taking the derivative of $G'(t)$ in respect to t, one has

$$G''(t) = -4b(t)\int_0^\pi w_x(x,t)w_{xt}(x,t)dx.$$

By simple computation and using the integral in parts, we get

$$\begin{aligned} G''(t) &= 4b(t)\int_0^\pi w_{xx}(x,t)w_t(x,t)dx \\ &= 4b^2(t)\int_0^\pi w_x^2(x,t)dx. \end{aligned} \tag{11}$$

Now, from (8) and applying the Holder inequality, we have

$$\begin{aligned} \int_0^\pi w_x^2(x,t)dx &= -\int_0^\pi w(x,t)w_{xx}(x,t)dx \\ &\le \left(\int_0^\pi w^2(x,t)dx\right)^{\frac{1}{2}}\left(\int_0^\pi w_{xx}^2(x,t)dx\right)^{\frac{1}{2}}. \end{aligned} \tag{12}$$

Thus (7)-(8)-(9) imply

$$(G'(t))^2 \le G(t)G''(t).$$

Hence by the Theorem 11 [5], p.65, which gives $G(t) = 0$. This implies that $u(x,t) = v(x,t)$. The proof is completed.

Step 2. The Problem (1) has a solution which is defined in (8).

To prove this, we only check that u satisfy three equations in system (1). By taking the derivative of u, we have

$$\begin{aligned} u_t &= \sum_{p=1}^\infty -p^2 b(t)\exp\left(p^2\int_t^T b(\xi)d\xi\right) g_p\sin(px) \\ &= \sum_{p=1}^\infty -p^2 b(t) < u(x,t), \sin px > \sin(px) \\ &= b(t)u_{xx}. \end{aligned}$$

In spite of the uniqueness, the problem (1) is still illposed and some regularization methods are necessary. In next Section, we introduce the approximation problem.

3. Regularization by a mollification method

In this section, we shall regularize the problem (1) by pertubing the final value g with a new way. Motivated by the idea of Clark and Oppenheimer [3], we approximate problem by the following problem

$$\begin{cases} u_t^\epsilon - b(t)u_{xx}^\epsilon = 0, \ (x,t)\in(0,\pi)\times(0,T), \\ u^\epsilon(0,t) = u^\epsilon(\pi,t) = 0, t\in(0,T) \\ u^\epsilon(x,T) = \sum_{p=1}^\infty \dfrac{\exp\{-p^2\int_0^T b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p\sin(px), x\in(0,\pi). \end{cases} \tag{13}$$

where $0 < \epsilon < 1$,

$$f_p(t) = \frac{2}{\pi}\int_0^\pi f(x,t)\sin(px)dx, \quad g_p = \frac{2}{\pi}\int_0^\pi g(x)\sin(px)dx$$

and $< .,. >$ is the inner product in $L^2((0,\pi))$. Note that if $b(t) = 1$ and $k = 0$ then the problem (1) has been considered in [3].
We shall prove that, the (unique) solution u^ϵ of (11) satisfies the following equality

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p \sin(px). \tag{14}$$

Lemma 3.1
For $M, \epsilon, x > 0$, $k \geq 1$, we have the inequality

$$\frac{1}{\epsilon x^k + e^{-Mx}} \leq (kM)^k \epsilon^{-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{-k}.$$

Proof.
Let the function f defined by $f(x) = \frac{1}{\epsilon x^k + e^{-Mx}}$. By taking the derivative of f, one has

$$f'(x) = \frac{\epsilon k x^{k-1} - Me^{-Mx}}{-(\epsilon x^k + e^{-Mx})^2}.$$

The equation $f'(x) = 0$ gives a unique solution x_0 such that $\epsilon k x_0^{k-1} - Me^{-Mx_0} = 0$. It means that $x_0^{k-1}e^{Mx_0} = \frac{M}{k\epsilon}$. Thus the function f achieves its maximum at a unique point $x = x_0$. Thus

$$f(x) \leq \frac{1}{\epsilon x_0^k + e^{-Mx_0}}.$$

Since $e^{-Mx_0} = \frac{k\epsilon}{M}x_0^{k-1}$, one has

$$f(x) \leq \frac{1}{\epsilon x_0^k + e^{-Mx_0}} \leq \frac{1}{\epsilon x_0^k + \frac{k\epsilon}{M}x_0^{k-1}}.$$

By using the inequality $e^{Mx_0} \geq Mx_0$, we get

$$\begin{aligned}
\frac{M}{k\epsilon} &= x_0^{k-1}e^{Mx_0} \\
&\leq \frac{1}{M^{k-1}}e^{(k-1)Mx_0}e^{Mx_0} \\
&= \frac{1}{M^{k-1}}e^{kMx_0}.
\end{aligned}$$

This gives $e^{kMx_0} \geq \frac{M^k}{k\epsilon}$ or $kMx_0 \geq \ln(\frac{M^k}{k\epsilon})$. Therefore

$$x_0 \geq \frac{1}{kM}\ln(\frac{M^k}{k\epsilon}).$$

Hence, we obtain

$$f(x) \leq \frac{1}{\epsilon x_0^k} \leq \frac{(kM)^k}{\epsilon \ln^k(\frac{M^k}{k\epsilon})}.$$

Lemma 3.2
For $0 \leq m \leq M$, we have the following inequality

$$\frac{e^{-mx}}{\epsilon x^k + e^{-Mx}} \leq (kM)^k \epsilon^{\frac{m}{M}-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{k(\frac{m}{M}-1)}.$$

Proof.
Since the inequality $\frac{1}{\epsilon x^k + e^{-Mx}} \leq (kM)^k \epsilon^{-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{-k}$, we obtain

$$\begin{aligned}
\frac{e^{-mx}}{\epsilon x^k + e^{-Mx}} &= \frac{e^{-mx}}{(\epsilon x^k + e^{-Mx})^{\frac{m}{M}}(\epsilon x^k + e^{-Mx})^{1-\frac{m}{M}}} \\
&\leq \frac{1}{(\epsilon x^k + e^{-Mx})^{1-\frac{m}{M}}} \\
&\leq \left[(kM)^k \epsilon^{-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{-k}\right]^{1-\frac{m}{M}} \\
&\leq (kM)^{k(1-\frac{m}{M})} \epsilon^{\frac{m}{M}-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{k(\frac{m}{M}-1)} \\
&\leq (kM)^k \epsilon^{\frac{m}{M}-1}\left(\ln(\frac{M^k}{k\epsilon})\right)^{k(\frac{m}{M}-1)}.
\end{aligned}$$

In next Theorem, we shall study the existence, the uniqueness and the stability of a (weak) solution of Problem (6)-(8). In fact, one has

Theorem 3.1 *The problem (11) has uniquely a weak solution $u^\epsilon \in$ satisfying (10). The solution depends continuously on g in $L^2(0,\pi)$).*

Proof

The proof is divided into two steps. In Step 1, we prove the existence and the uniqueness of a solution of (6)-(8). In Step 2, the stability of the solution is given. Denote $W = ([0,T]; L^2(0,\pi) \cap L^2(0,T; H_0^1(0,\pi)) \cap C^1(0,T; H_0^1(0,\pi))$.

Step 1. The existence and the uniqueness of a solution of (6)-(8)

We divide this step into two parts.

Part A If $u^\epsilon \in W$ satisfies (11) then u^ϵ is solution of (6)-(8).
We have:

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p \sin(px). \tag{15}$$

We can verify directly that $u^\epsilon \in W$. Moreover, one has

$$\begin{aligned} < u_t^\epsilon(x,t), \sin(px) > &= \frac{-p^2 b(t)\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p \\ &= -p^2 b(t) < u^\epsilon(x,t), \sin px > \\ &= b(t) < u_{xx}^\epsilon(x,t), \sin(px). \end{aligned}$$

This implies that

$$u_t^\epsilon = b(t)u_{xx}^\epsilon$$

$$u^\epsilon(x,T) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2\int_0^T b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p \sin(px).$$

So u^ϵ is the solution of $(6)-(8)$.

Part B. The Problem (6)-(8) has at most one solution $C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$.

Proof.
We can prove this Theorem by similar way in Step 1 of Theorem 2.2.
Since Part A and Part B, we complete the proof of Step 1.

Step 2. The solution of the problem $(6)-(8)$ depends continuously on g in $L^2(0,\pi)$.
Let u and v be two solutions of $(6)-(8)$ corresponding to the final values g and h.
From we have

$$u(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} g_p \sin(px). \tag{16}$$

$$v(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}} h_p \sin(px), \tag{17}$$

where

$$\begin{aligned} g_p &= \frac{2}{\pi}\int_0^\pi g(x)\sin(px)dx, \\ h_p &= \frac{2}{\pi}\int_0^\pi h(x)\sin(px)dx. \end{aligned}$$

This follows that

$$\begin{aligned} u(.,t) - v(.,t)^2 &= \frac{\pi}{2}\sum_{p=1}^{\infty}\left|\frac{\exp\{-p^2\int_0^t b(\xi)d\xi\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}}(g_p - h_p)\right|^2, \\ &\leq \frac{\pi}{2}\sum_{p=1}^{\infty}\left|\frac{\exp\{-B_1 t p^2\}}{\epsilon p^{2k} + \exp\{-B_2 T p^2\}}(g_p - h_p)\right|^2, \\ &\leq \frac{\pi}{2}\left((kB_2T)^k \epsilon^{\frac{B_1 t}{B_2 T}-1}\left(\ln(\frac{(B_2T)^k}{k\epsilon})\right)^{k(\frac{B_1 t}{B_2 T}-1)}\right)^2 \sum_{p=1}^{\infty}|g_p - h_p|^2 \\ &= B_4^2 \epsilon^{\frac{2B_1 t}{B_2 T}-2}\left(\ln(\frac{B_3}{\epsilon})\right)^{-2k(1-\frac{B_1 t}{B_2 T})}\|g-h\|^2. \end{aligned} \tag{18}$$

where

$$B_3 = \frac{(B_2T)^k}{k}, \tag{19}$$

$$B_4 = (kB_2T)^k. \tag{20}$$

Hence

$$u(.,t) - v(.,t) \leq B_4 \epsilon^{\frac{B_1 t}{B_2 T}-1}\left(\ln(\frac{B_3}{\epsilon})\right)^{-k(1-\frac{B_1 t}{B_2 T})} g - h.$$

This completes the proof of Step 2 and the proof of our theorem.

Theorem 3.2 *Let $g(x) \in L^2(0,\pi)$ be the function satisfies the following condition*

$$\sum_{p=1}^{\infty} p^{4k} e^{2TB_2p^2} g_p^2 < \infty,$$

where $g_p = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(px)dx$. Then $u^{\epsilon}(x,T)$ converges to $g(x)$ in $L^2(0,\pi)$ with order $\left(\ln(\frac{B_3}{\epsilon})\right)^{-k}$ as ϵ tends to zero.

Proof.

We have $g(x) = \sum\limits_{p=1}^{\infty} g_p \sin(px)$, where g_p is defined in (9). Let $\alpha > 0$. Then there exists a positive integer number N for which $\frac{\pi}{2}\sum\limits_{p=N+1}^{\infty} g_p^2 < \alpha/2$. We have

$$\|u^{\epsilon}(x,T) - g(x)\|^2 = \frac{\pi}{2}\sum_{p=1}^{\infty}\frac{\epsilon^2 p^{4k} g_p^2}{\left(\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi\}\right)^2}. \tag{21}$$

Then since

$$\begin{aligned}\left(\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi)d\xi\}\right)^2 &> \epsilon^2 p^{4k} + \exp\{-2p^2 \int_0^T b(\xi)d\xi\} \\ &> \epsilon^2 p^{4k} + e^{-2TB_2 p^2} \\ &> e^{-2TB_2 p^2},\end{aligned}$$

we get

$$\|u^\epsilon(x,T) - g(x)\|^2 \le \epsilon^2 \frac{\pi}{2} \sum_{p=1}^{N} p^{4k} g_p^2 e^{2TB_2 p^2} + \frac{\alpha}{2}.$$

By taking ϵ such that $\epsilon < \sqrt{\alpha}\left(\pi \sum\limits_{p=1}^{N} p^{4k} g_p^2 e^{2TB_2 p^2}\right)^{\frac{-1}{2}}$, we get

$$\|u^\epsilon(x,T) - g(x)\|^2 < \alpha.$$

We end the proof.
By using the inequality

$$\begin{aligned}\frac{1}{\epsilon x^k + e^{-B_2 T x}} &\le (kTB_2)^k \epsilon^{-1} \left(\ln(\frac{(B_2 T)^k}{k\epsilon})\right)^{-k} \\ &= B_4 \epsilon^{-1} \left(\ln(\frac{(B_3)}{\epsilon})\right)^{-k}\end{aligned}$$

we have the error estimate

$$\begin{aligned}\|u^\epsilon(x,T) - g(x)\|^2 &= \frac{\pi}{2} \sum_{p=1}^{\infty} \frac{\epsilon^2 p^{4k} g_p^2}{\left(\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi)d\xi\}\right)^2} \\ &\le \frac{\pi}{2} \sum_{p=1}^{\infty} \frac{\epsilon^2 p^{4k} g_p^2}{\left(\epsilon p^{2k} + e^{-TB_2 p^2}\right)^2} \\ &\le \frac{\pi}{2} \epsilon^2 B_4^2 \epsilon^{-2} \left(\ln(\frac{(B_3)}{\epsilon})\right)^{-2k} \sum_{p=1}^{\infty} p^{4k} g_p^2 \\ &= B_4^2 \left(\ln(\frac{B_3}{\epsilon})\right)^{-2k} \frac{\pi}{2} \sum_{p=1}^{\infty} p^{4k} g_p^2.\end{aligned}$$

This implies that

$$\|u^\epsilon(x,T) - g(x)\| \le B_4 \left(\ln(\frac{B_3}{\epsilon})\right)^{-k} \sqrt{\frac{\pi}{2} \sum_{p=1}^{\infty} p^{4k} g_p^2}.$$

This ends the proof.

Theorem 3.3

Let $g \in L^2(0,\pi)$ be as Theorem 3.2. If $u^\epsilon(x,0)$ converges in $L^2(0,\pi)$, then the problem (1)-(3) has a unique solution u. Furthermore, the regularized solution $u^\epsilon(x,t)$ converges to $u(t)$ as ϵ tends to zero uniformly in t.

Proof. Assume that $\lim_{\epsilon\to 0} u^\epsilon(x,0) = u_0(x)$ exists. Let

$$u(x,t) = \sum_{p=1}^{\infty} \exp\{-p^2 \int_0^t b(\xi)d\xi\} u_{0p} \sin(px)$$

where $u_{0p} = \frac{2}{\pi}\int_0^\pi u_0(x)\sin(px)dx$.

It is clear to see that $u(x,t)$ satisfies (1)-(2). We have the formula of $u^\epsilon(x,t)$

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty} \exp\{-p^2 \int_0^t b(\xi)d\xi\} u^\epsilon_{0p} \sin(px)$$

where $u^\epsilon_{0p} = \frac{2}{\pi}\int_0^\pi u^\epsilon(x,0)\sin(px)dx$.

We have in view of the inequality $(a+b)^2 \le 2(a^2+b^2)$

$$\begin{aligned} \|u^\epsilon(.,t) - u(.,t)\|^2 &\le \frac{\pi}{2}\sum_{p=1}^{\infty} \exp\left(-2p^2\int_0^t b(s)ds\right)(u^\epsilon_{0p} - u_{0p})^2 \\ &\le \|u^\epsilon(.,0) - u_0(.)\|^2. \end{aligned}$$

Hence $\lim_{\epsilon\to 0} u^\epsilon(x,t) = u(x,t)$. Thus $\lim_{\epsilon\to 0} u^\epsilon(x,T) = u(x,T)$. Using the theorem 2.2, we have $u(x,T) = g(x)$. Hence, $u(x,t)$ is the unique solution of the problem (1)-(3). We also see that $u^\epsilon(x,t)$ converges to $u(x,t)$ uniformly in t.

Theorem 3.4

Let $f \in L^2(0,T;L^2(0,\pi))$ and $g \in L^2(0,\pi)$. Suppose Problem (1)-(3) has a unique solution $u(x,t)$ in $C([0,T];H^1_0(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$ which satisfies $\sum_{p=1}^{\infty} p^{4k}u_p^2(t) < \infty$. Then

$$u(.,t) - u^\epsilon(.,t) \le C\left(\ln(\frac{B_3}{\epsilon})\right)^{-k}$$

for every $t \in [0,T]$, where $C = B_4\sqrt{\frac{\pi}{2}\sup_{t\in[0,T]}\sum_{p=1}^{\infty} p^{4k}u_p^2(t)}$ and u^ϵ is the unique solution of Problem (6)-(8).

Proof
Suppose the Problem (1)-(3) has an exact solution $u \in C([0,T]; H_0^1(0,\pi)) \cap C^1((0,T); L^2(0,\pi))$, we get the following formula

$$u(x,t) = \sum_{p=1}^{\infty} \left[\exp\left(p^2 \int_t^T b(\xi) d\xi \right) g_p \right] \sin(px). \tag{22}$$

Since (10) and (20), we get

$$\begin{aligned} & |u_p(t) - u_p^\epsilon(t)| = \\ = & \left| \left[\exp\left(p^2 \int_t^T b(\xi) d\xi \right) - \frac{\exp\{-p^2 \int_0^t b(\xi) d\xi\}}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi) d\xi\}} \right] \right| g_p \\ = & \left| \epsilon p^{2k} \frac{\exp\left(p^2 \int_t^T b(\xi) d\xi \right)}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi) d\xi\}} \right| g_p \\ \le & \ \epsilon \frac{1}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi) d\xi\}} \left| p^{2k} \exp\left(p^2 \int_t^T b(\xi) d\xi \right) g_p \right| \\ \le & \ \frac{\epsilon}{\epsilon p^{2k} + e^{-B_2 T p^2}} \left| p^{2k} \exp\left(p^2 \int_t^T b(\xi) d\xi \right) g_p \right| . \end{aligned}$$

This follows that

$$\begin{aligned} & u(.,t) - u^\epsilon(.,t)^2 = \frac{\pi}{2} \sum_{p=1}^{\infty} |u_p(t) - u_p^\epsilon(t)|^2 \\ & \le \frac{\pi}{2} B_4^2 \left(\ln(\frac{B_3}{\epsilon}) \right)^{-2k} \sum_{p=1}^{\infty} p^{4k} u_p^2(t) \\ & \le C^2 \left(\ln(\frac{B_3}{\epsilon}) \right)^{-2k} . \end{aligned}$$

Hence

$$u(.,t) - u^\epsilon(.,t) \le C \left(\ln(\frac{B_3}{\epsilon}) \right)^{-k}$$

where $C = B_4 \sqrt{\frac{\pi}{2} \sup_{t \in [0,T]} \sum_{p=1}^{\infty} p^{4k} u_p^2(t)}$.
This completes the proof of Theorem.

Theorem 3.5 *Let f, g, ϵ be as in Theorem 3.4. Assume that the exact solution u of (1)–(3) corresponding to g satisfies $u \in W$ and $\sum_{p=1}^{\infty} p^{4k} u_p^2(t) < \infty$. Let $g_\epsilon \in L^2(0,\pi)$ be a measured data such that $g_\epsilon - g \le \epsilon$. Then there exists a function v^ϵsatisfying*

$$u(.,t) - v^\epsilon(.,t) \le \left[C + \left(\epsilon \ln^k (\frac{B_3}{\epsilon}) \right)^{\frac{B_1 t}{B_2 T}} \right] \left(\ln(\frac{B_3}{\epsilon}) \right)^{-k}$$

for every $t \in [0,T]$ and C is defined in theorem 3.4.

Proof

Let u^ϵ be the solution of problem (6)-(8) corresponding to g and let v^ϵ be the solution of problem (6)-(8) corresponding to g_ϵ where g, g_ϵ are in right hand side of (6). Using Theorem 3.4 and Step 2 of theorem 3.1, we get

$$\begin{aligned} v^\epsilon(.,t) - u(.,t) &\le v^\epsilon(.,t) - u^\epsilon(.,t) + u^\epsilon(.,t) - u(.,t) \\ &\le \epsilon^{\frac{B_1 t}{B_2 T} - 1} \left(\ln(\frac{B_3}{\epsilon}) \right)^{-k(1 - \frac{B_1 t}{B_2 T})} g_\epsilon - g + C \left(\ln(\frac{B_3}{\epsilon}) \right)^{-k} \\ &\le \left(\ln(\frac{B_3}{\epsilon}) \right)^{-k} \left[C + \left(\epsilon \ln^k (\frac{B_3}{\epsilon}) \right)^{\frac{B_1 t}{B_2 T}} \right] \end{aligned}$$

for every $t \in (0,T)$ and where C is defined in Theorem 3.4

This completed the proof of Theorem.

4. Improved estimates with Hölder type

In Section 3, Theorem 3.4, with the condition $\sum_{p=1}^{\infty} p^{4k} u_p^2(t) < \infty$, we establish the error between the exact solution and regularized solution which is of logarithmic order. The convergence rates here are very slow. To get a stability estimate of Hölder type for the whole $[0;T]$, we introduce a new regularized problem, which is given by

$$\begin{cases} u_t^\epsilon - b(t) u_{xx}^\epsilon = 0, \ (x,t) \in (0,\pi) \times (0,T) \\ \mathrm{u}^\epsilon(0,t) = u^\epsilon(\pi,t) = 0, t \in (0,T) \\ \mathrm{u}^\epsilon(x,T) = \sum_{p=1}^{\infty} \dfrac{\exp\{-p^2 \int_0^T b(\xi) d\xi - mp^2\}}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi) d\xi - mp^2\}} g_p \sin(px), x \in (0,\pi) \end{cases} \quad (23)$$

where $m \ge 0$ is a fixed number and $0 < \epsilon < 1$. If $m = 0$ then the problem (23) is also the problem (13) which introduced in Section 3. The (unique) solution u^ϵ of

(23) satisfies the following equality

$$u^\epsilon(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2 \int_0^t b(\xi)d\xi - mp^2\}}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi)d\xi - mp^2\}} g_p \sin(px), \quad 0 \le t \le T. \tag{24}$$

Let v^ϵ be the solution of (23) corresponding to the measured data g^ϵ. Then we also have

$$v^\epsilon(x,t) = \sum_{p=1}^{\infty} \frac{\exp\{-p^2 \int_0^t b(\xi)d\xi - mp^2\}}{\epsilon p^{2k} + \exp\{-p^2 \int_0^T b(\xi)d\xi - mp^2\}} g_p^\epsilon \sin(px), \quad 0 \le t \le T. \tag{25}$$

where $g_p^\epsilon = \frac{2}{\pi}\int_0^\pi g^\epsilon(x)\sin pxdx$. The main purpose of this section is of considering the error $\|v^\epsilon(.,t) - u(.,t)\|$.
We have the following theorem
Theorem 4.1
Let $f \in L^2(0,T;L^2(0,\pi))$ and $g \in L^2(0,\pi)$. Suppose Problem (1)-(3) has a unique solution $u(x,t)$ in $C([0,T];H_0^1(0,\pi)) \cap C^1((0,T);L^2(0,\pi))$ which satisfies

$$\sum_{p=1}^{\infty} p^{4k} e^{2mp^2} u_p^2(0) < \infty, \tag{26}$$

where $u_p(0) = \frac{2}{\pi}\int_0^\pi u(x,0)\sin pxdx$. Then the following estimate is holds

$$u(.,t) - v^\epsilon(.,t) \le E(m,k)\epsilon^{\frac{B_1t+m}{B_2T+m}} \left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{B_1t+m}{B_2T+m}-1\right)}. \tag{27}$$

for every $t \in [0,T]$, where

$$\begin{aligned} C(m,k) &= (kB_2T + km)^k \\ D(m,k) &= \frac{(B_2T+m)^k}{k} \\ E(m,k) &= C(m,k)\Big(\sqrt{\sum_{p=1}^{\infty} p^{4k} e^{2mp^2} u_p^2(0)} + 1\Big). \end{aligned}$$

Remark.
1. If $m = 0$, then error $u(.,t) - v^\epsilon(.,t)$ is of order

$$\epsilon^{\frac{B_1t}{B_2T}} \left(\ln(\frac{D(0,k)}{\epsilon})\right)^{k\left(\frac{B_1t}{B_2T}-1\right)}.$$

Let $t = 0$ then $u(.,0) - v^\epsilon(.,0)$ is of order $\left(\ln(\frac{D(0,k)}{\epsilon})\right)^{-k}$, which is the same order as one in Theorem 3.4.

2.The error (27) is of order $\epsilon^{\frac{m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{m}{B_2T+m}-1\right)}$ for all $t \in [0,T]$. As we know, the convergence rate of $\epsilon^{\frac{m}{B_2T+m}}$ is faster than that of the logarithmic order $\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{m}{B_2T+m}-1\right)}$ which is introduced in Section 3. To our knowledge, this seems to be the optimal error order for backward heat. This is a strong point of this method.

Proof.

Step 1. We estimate $\|u^\epsilon(.,t) - u(.,t)\|$. We have

$$\begin{aligned}
& u_p(t) - u_p^\epsilon(t) = \\
= & \left[\exp\left(p^2\int_t^T b(\xi)d\xi\right) - \frac{\exp\{-p^2\int_0^t b(\xi)d\xi - mp^2\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}}\right] g_p \\
= & \epsilon p^{2k}\frac{\exp\left(p^2\int_t^T b(\xi)d\xi\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} g_p \\
= & \epsilon p^{2k}\frac{\exp\left(p^2\int_t^T b(\xi)d\xi\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} g_p. \qquad (28)
\end{aligned}$$

Since $< u(x,t), \sqrt{\frac{2}{\pi}}\sin px >= \exp\left(p^2\int_t^T b(\xi)d\xi\right) g_p$, we get

$$u_p(0) = \frac{2}{\pi} < u(x,0), \sin px >= \exp\left(p^2\int_0^T b(\xi)d\xi\right) g_p,$$

Or

$$g_p = \exp\left(-p^2\int_0^T b(\xi)d\xi\right) u_p(0). \qquad (29)$$

Combining (28) and (29), we obtain

$$\begin{aligned}
u_p(t) - u_p^\epsilon(t) &= \epsilon p^{2k}\frac{\exp\left(-p^2\int_0^t b(\xi)d\xi\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} u_p(0) \\
&= \epsilon p^{2k}\frac{\exp\left(-p^2\int_0^t b(\xi)d\xi - mp^2\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} \exp\left(mp^2\right) u_p(0). \qquad (30)
\end{aligned}$$

On the other hand, we note that

$$\begin{aligned}\frac{\exp\left(-p^2\int_0^t b(\xi)d\xi - mp^2\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} &\le \frac{e^{-(B_1t+m)p^2}}{\epsilon p^{2k} + e^{-(B_2T+m)p^2}} \\ &\le (kB_2T+km)^k \epsilon^{\frac{B_1t+m}{B_2T+m}-1}\left(\ln(\frac{(B_2T+m)^k}{k\epsilon})\right)^{k(\frac{B_1t+m}{B_2T+m}-1)} \quad (31)\end{aligned}$$

For a short, we denote

$$\begin{aligned} C(m,k) &= (kB_2T+km)^k \\ D(m,k) &= \frac{(B_2T+m)^k}{k}. \end{aligned}$$

Then (31) can be rewritten as follows

$$\frac{\exp\left(-p^2\int_0^t b(\xi)d\xi - mp^2\right)}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}} \le C(m,k)\epsilon^{\frac{B_1t+m}{B_2T+m}-1}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{B_1t+m}{B_2T+m}-1\right)}. \quad (32)$$

It follows from (30) and (32), we get

$$\begin{aligned} &\|u(.,t) - u^\epsilon(.,t)\|^2 \\ &= \sum_{p=1}^\infty |u_p(t) - u_p^\epsilon(t)|^2 \\ &\le \epsilon^2 C^2(m,k)\epsilon^{2\frac{B_1t+m}{B_2T+m}-2}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)} \sum_{p=1}^\infty p^{4k}\exp\left(2mp^2\right)u_p^2(0) \\ &\le C^2(m,k)\epsilon^{2\frac{B_1t+m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)} \sum_{p=1}^\infty p^{4k}e^{2mp^2}u_p^2(0). \end{aligned}$$

Therefore

$$\begin{aligned} &\|u(.,t) - u^\epsilon(.,t)\| \\ &\quad \le C(m,k)\epsilon^{\frac{B_1t+m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{B_1t+m}{B_2T+B_2m}-1\right)}\sqrt{\sum_{p=1}^\infty p^{4k}e^{2mp^2}u_p^2(0)}. \quad (33) \end{aligned}$$

Step 2. We estimate the term $\|v^\epsilon(.,t) - u^\epsilon(.,t)\|$. Indeed, we have

$$\begin{aligned}\|v^\epsilon(.,t) - u^\epsilon(.,t)\|^2 &= \sum_{p=1}^{\infty}\Big(\frac{\exp\{-p^2\int_0^t b(\xi)d\xi - mp^2\}}{\epsilon p^{2k} + \exp\{-p^2\int_0^T b(\xi)d\xi - mp^2\}}\Big)^2\Big(g_p^\epsilon - g_p\Big)^2 \\ &\le \sum_{p=1}^{\infty}\Big(\frac{e^{-(B_1t+m)p^2}}{\epsilon p^{2k} + e^{-(B_2T+m)p^2}}\Big)^2\Big(g_p^\epsilon - g_p\Big)^2.\end{aligned}$$

Using (32) we obtain

$$\begin{aligned}&\|v^\epsilon(.,t) - u^\epsilon(.,t)\|^2 \\ \le\ & C^2(m,k)\epsilon^{2\frac{B_1t+m}{B_2T+m}-2}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)}\sum_{p=1}^{\infty}\Big(g_p^\epsilon - g_p\Big)^2 \\ \le\ & C^2(m,k)\epsilon^{2\frac{B_1t+m}{B_2T+m}-2}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)}\|g^\epsilon - g\|^2 \\ \le\ & C^2(m,k)\epsilon^{2\frac{B_1t+m}{B_2T+m}-2}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)}\epsilon^2 \\ =\ & C^2(m,k)\epsilon^{\frac{2B_1t+2m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{2k\left(\frac{B_1t+m}{B_2T+m}-1\right)}.\end{aligned}$$

Hence

$$\|v^\epsilon(.,t) - u^\epsilon(.,t)\| \le C(m,k)\epsilon^{\frac{B_1t+m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{B_1t+m}{B_2T+m}-1\right)}. \tag{34}$$

Combining (33) and (34), we obtain

$$\begin{aligned}&\|u^\epsilon(.,t) - u(.,t)\| \le \|v^\epsilon(.,t) - u^\epsilon(.,t)\| + \|u^\epsilon(.,t) - u(.,t)\| \\ &\quad\le C(m,k)\epsilon^{\frac{B_1t+m}{B_2T+m}}\left(\ln(\frac{D(m,k)}{\epsilon})\right)^{k\left(\frac{B_1t+m}{B_2T+m}-1\right)}\Big(\sqrt{\sum_{p=1}^{\infty}p^{4k}e^{2mp^2}u_p^2(0)} + 1\Big).\end{aligned}$$

5.Conclusion

We have considered a regularization problem for an initial inverse heat equation with time-dependent coefficient, namely Problem (1). We also establish the error estimate of H"older type for all $t \in [0,T]$. This estimate improves the results in many earlier works. In the future work, we will consider the regularized problem for the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big(b(x,t)\dfrac{\partial u}{\partial x}\Big), (x,t) \in (0,\pi) \times (0,T) \\ u(0,t) = u(\pi,t) = 0, t \in (0,T) \\ u(x,T) = g(x), (x,t) \in (0,\pi) \times (0,T) \end{cases} \tag{35}$$

where $b(x,t)$ is a function dependent on both variables x, t.

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