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APPLICATIONS OF SLOWLY CHANGING FUNCTIONS IN THE ESTIMATION OF GROWTH RATES OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using L^*-order and L^*-type and differential monomials, differential polynomials generated by one of the factors.

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Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [15]. In the sequel we use the following notation : $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Let f be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, ... n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} ... \left(f^{(k)}\right)^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{Mj} = \sum\limits_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[4],[10]} . The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1\leq\, j\leq\, s} \gamma_{Mj}$ and $\Gamma_P = \max\limits_{1\leq\, j\leq\, s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$ {[4],[10]}. Also we call the numbers $\underline{\gamma_P} = \min\limits_{1\leq\, j\leq\, s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e., for which $n_{0j} = 0$ for $j = 1, 2, ...s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

In the sequel the following definitions are well known :

Definition 1. Let ' a ' be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of ' a ' with respect to a meromorphic function f are defined as

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}$$

and

$$\Delta(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}.$$

Definition 2. The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

Definition 3. [14] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r, a; f \mid= 1)$, the number of simple zeros of $f - a$ in $|z| \leq r$. $N(r, a; f \mid= 1)$ is defined in terms of $n(r, a; f \mid= 1)$ in the usual way. We put

$$\delta_1(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f \mid= 1)}{T(r, f)} \ ,$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f - a$.

Yang [13]proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and ${}_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a; f) \leq 4$.

Definition 4. [8] For $a \epsilon \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)}.$$

Definition 5. [3] $P[f]$ is said to be admissible if
(i) $P[f]$ is homogeneous, or
(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$.

Now let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [11] defined it in the following way :

Definition 6.[11]A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon (> 0)$,

$$\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \; for \; r \geq r(\varepsilon) \text{ and}$$

uniformly for $k (\geq 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \; .$$

Somasundaram and Thamizharasi [12] introduced the notions of L-order and L-order for entire functions. The more generalised concept for L-order and L-type for entire and meromorphic functions are L^*-order and L^*-type respectively. Their definitions are as follows :

Definition 7. [12] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log\left[re^{L(r)}\right]} \; .$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log\left[re^{L(r)}\right]}.$$

Definition 8. [12] The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as follows:

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \ , \ 0 < \rho_f^{L^*} < \infty \ .$$

For meromorphic f,

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \ , \ 0 < \rho_f^{L^*} < \infty \ .$$

Lakshminarasimhan [6] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [7] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In the paper we investigate the comparative growth of composite entire and meromorphic functions and differential monomials, differential polynomials generated by their factors using L^*-order and L^*-type. It is needless to mention that the admissibility and homogenity of $P_0[f]$ will be required as per the requirements of the lemmas and theorems in the paper.

2.Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [1] If f be meromorphic and g be entire then for all sufficiently large values of r,

$$T(r, f\circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f) \ .$$

Lemma 2. [2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f\circ g) \geq T(\exp(r)^{\mu}, f) \ .$$

Lemma 3. [3] Let $P_0[f]$ be admissible. If f is of finite order or of non zero lower order and $\sum_{a\neq\infty} \Theta(a; f) = 2$, then

$$\lim_{r\to\infty} \frac{T(r, P_0[f])}{T(r,f)} = \Gamma_{P_0}.$$

Lemma 4. [3] Let f be either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$. Then for homogeneous $P_0[f]$,

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\gamma_{P_0}.$$

Lemma 5. Let f be a meromorphic function of finite order or of non zero lower order. If $\sum\limits_{a\neq\infty}\Theta(a;f)=2$, then the L^*-order (L^*-lower order) of admissible $P_0[f]$ is same as that of f.

Proof. By Lemma 3, $\lim\limits_{r\to\infty}\frac{\log T(r,P_0[f])}{\log T(r,f)}$ exists and is equal to 1.

$$\begin{aligned}\rho^{L^*}_{P_0[f]} &= \limsup_{r\to\infty}\frac{\log T(r,P_0[f])}{\log\left[re^{L(r)}\right]}\\ &= \limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.\lim_{r\to\infty}\frac{\log T(r,P_0[f])}{\log T(r,f)}\\ &= \rho^{L^*}_f.1\\ &= \rho^{L^*}_f.\end{aligned}$$

In a similar manner, $\lambda^{L^*}_{P_0[f]}=\lambda^{L^*}_f$.
This proves the lemma.

Lemma 6. Let f be a meromorphic function of finite order or of non zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$. Then the L^*-order (L^*-lower order) of homogeneous $P_0[f]$ and f are same.

We omit the proof of the lemma because it can be carried out in the line of Lemma 5 and with the help of Lemma 4.

Lemma 7. [9] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and ${}_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Then

$$\lim_{r\to\infty}\frac{T(r,M[f])}{T(r,f)}=\Gamma_M-(\Gamma_M-\gamma_M)\Theta(\infty;f)\ ,$$

where

$$\Theta(\infty;f)=1-\limsup_{r\to\infty}\frac{\bar{N}(r,f)}{T(r,f)}\ .$$

Lemma 8. If f be a transcendental meromorphic function of finite order or of non-zero lower order and $_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$, then the L^*-order (L^*-lower order) of $M[f]$ are same as those of f.

We omit the proof of the lemma because it can be carried out in the line of Lemma 5 and with the help of Lemma 7.

3. Theorems

In this section we present the main results of the paper.

It is needless to mention that in the paper, the admissibility and homogenity of $P_0[f]$ will be needed as per the requirements of the theorems.

Theorem 1. Let f be meromorphic with finite order or non zero lower order and g be entire satisfying the following conditions:
(i) $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and (ii) $\sum\limits_{a\neq\infty}\Theta(a;f) = 2$. Then for any $A > 0$

$$\limsup_{r\to\infty}\frac{\log^{[2]} T\left(\exp\left(r^A\right), f\circ g\right)}{\log T\left(\exp\left(r^\mu\right), P_0[f]\right) + K(r,g;L)} = \infty\ ,$$

where $0 < \mu < \rho_g$ and $K(r,g;L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise .}\end{cases}$

Proof. Let $0 < \mu < \mu' < \rho_g$. Using the definition of L^*-lower order we obtain in view of Lemma 2 for a sequence of values of r tending to infinity that

$$\log T\left(\exp\left(r^A\right), f\circ g\right) \geq \log T\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}, f\right)$$

$$\begin{aligned} &i.e.,\ \log T\left(\exp\left(r^A\right), f\circ g\right) \\ \geq\ &\left(\lambda_f^{L^*} - \varepsilon\right).\log\left\{\exp\left(\exp\left(r^A\right)\right)^{\mu'}.\exp L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\}\end{aligned}$$

$$\begin{aligned} &i.e.,\ \log T\left(\exp\left(r^A\right), f\circ g\right) \\ \geq\ &\left(\lambda_f^{L^*} - \varepsilon\right).\left\{\left(\exp\left(r^A\right)\right)^{\mu'} + L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\}\end{aligned}$$

$$\begin{aligned} &i.e.,\ \log T\left(\exp\left(r^A\right), f\circ g\right) \\ \geq\ &\left(\lambda_f^{L^*} - \varepsilon\right).\left\{\left(\exp\left(r^A\right)\right)^{\mu'}\left(1 + \frac{L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right)\right\}\end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) &\geq O(1) + \mu' \log\exp\left(r^{A}\right) \\ &+ \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\mu'}\right)}{\left(\exp\left(r^{A}\right)\right)^{\mu'}}\right\} \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) \geq O(1) + \mu' r^{A} \\ + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{A}\right)\right)^{\mu'}\right)}{\left(\exp\left(r^{A}\right)\right)^{\mu'}}\right\} \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) \geq O(1) + \mu' r^{A} \\ + \log\left[1 + \frac{L\left(\exp\left(\exp\left(\mu' r^{A}\right)\right)\right)}{\exp\left(\mu' r^{A}\right)}\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) \geq O(1) + \mu r^{A} + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \\ - \log\left[\exp\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}\right] \\ + \log\left[1 + \frac{L\left(\exp\left(\exp\left(\mu' r^{A}\right)\right)\right)}{\exp\left(\mu r^{A}\right)}\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) \geq O(1) + \mu r^{A} + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \\ + \log\left[\frac{\exp\left(\mu r^{A}\right) + L\left(\exp\left(\exp\left(\mu' r^{A}\right)\right)\right)}{\exp\left(\mu r^{A}\right)\exp\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}}\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right) \geq O(1) + \mu' r^{(A-\mu)}.r^{\mu} \\ + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \ . \end{aligned} \quad (1)$$

Also in view of Lemma 5 we have for all sufficiently large values of r that

$$\log T\left(\exp\left(r^{\mu}\right), P_0[f]\right) \leq \left(\rho_{P_0[f]}^{L^*} + \varepsilon\right)\log\left\{\exp\left(r^{\mu}\right) e^{L\left(\exp\left(r^{\mu}\right)\right)}\right\}$$

$$i.e., \ \log T\left(\exp\left(r^{\mu}\right), P_0[f]\right) \leq \left(\rho_{f}^{L^*} + \varepsilon\right)\left\{\log\exp\left(r^{\mu}\right) + L\left(\exp\left(r^{\mu}\right)\right)\right\}$$

$$i.e., \ \log T\left(\exp\left(r^{\mu}\right), P_0[f]\right) \leq \left(\rho_{f}^{L^*} + \varepsilon\right)\left\{r^{\mu} + L\left(\exp\left(r^{\mu}\right)\right)\right\}$$

$$i.e., \ \frac{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) - \left(\rho_f^{L^*} + \varepsilon\right) L\left(\exp\left(r^{\mu}\right)\right)}{\left(\rho_f^{L^*} + \varepsilon\right)} \leq r^{\mu} \ . \tag{2}$$

Now from (1) and (2) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} &\log^{[2]} T\left(\exp\left(r^{A}\right), f \circ g\right) \\ &\geq O\left(1\right) + \left(\frac{\mu' r^{(A-\mu)}}{\rho_f^{L^*} + \varepsilon}\right) \left[\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) - \left(\rho_f^{L^*} + \varepsilon\right) L\left(\exp\left(r^{\mu}\right)\right)\right] \\ &+ L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \end{aligned} \tag{3}$$

$$\begin{aligned} i.e., \ \frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)} &\geq \frac{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) + O\left(1\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)} \\ &+ \frac{\mu' r^{(A-\mu)}}{\rho_f^{L^*} + \varepsilon} \left\{1 - \frac{\left(\rho_f^{L^*} + \varepsilon\right) L\left(\exp\left(r^{\mu}\right)\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)}\right\} \ . \end{aligned} \tag{4}$$

Again from (3) we get for a sequence of values of r tending to infinity that

$$\begin{aligned} &\frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} \\ &\geq \frac{O\left(1\right) - \mu' r^{(A-\mu)} L\left(\exp\left(r^{\mu}\right)\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} \\ &+ \frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_f^{L^*} + \varepsilon}\right) \log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} \\ &+ \frac{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} \end{aligned}$$

$$\begin{aligned} i.e., \ \frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f \circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} &\geq \frac{\frac{O(1) - \mu' r^{(A-\mu)} L(\exp(r^{\mu}))}{L(\exp(\exp(\mu r^{A})))}}{\frac{\log T(\exp(r^{\mu}), P_0[f])}{L(\exp(\exp(\mu r^{A})))} + 1} \\ &+ \frac{\frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_f^{L^*} + \varepsilon}\right) \log T(\exp(r^{\mu}), P_0[f])}{\log T(\exp(r^{\mu}), P_0[f])}}{1 + \frac{L(\exp(\exp(\mu r^{A})))}{\log T(\exp(r^{\mu}), P_0[f])}} + \frac{1}{1 + \frac{\log T(\exp(r^{\mu}), P_0[f])}{L(\exp(\exp(\mu r^{A})))}} \ . \end{aligned} \tag{5}$$

Case I. If $r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then it follows from (4) that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)} = \infty\ .$$

Case II. $r^{\mu} \neq o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\}$ then two sub cases may arise.
Sub case (a). If $L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) = o\left\{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)\right\}$, then we get from (5) that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty\ .$$

Sub case (b). If $L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \sim \log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)$ then

$$\lim_{r\to\infty}\frac{L\left\{\exp\left(\exp\left(\mu r^{A}\right)\right)\right\}}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right)} = 1$$

and we obtain from (5) that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)} = \infty\ .$$

Combining Case I and Case II we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[2]} T\left(\exp\left(r^{A}\right), f\circ g\right)}{\log T\left(\exp\left(r^{\mu}\right), P_0\left[f\right]\right) + K\left(r, g; L\right)} = \infty\ ,$$

where $K\left(r, g; L\right) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)\right\} \\ \hfill \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right) \text{ otherwise .}\end{cases}$

This proves the theorem.

Remark 1. With the help of Lemma 6, the conclusion of Theorem 1 can also be drawn under the hypothesis $\Theta\left(\infty; f\right) = \sum\limits_{a\neq\infty}\delta_p\left(a; f\right) = 1$ or $\delta\left(\infty; f\right) = \sum\limits_{a\neq\infty}\delta\left(a; f\right) = 1$ instead of $\sum\limits_{a\neq\infty}\Theta\left(a; f\right) = 2$.

Remark 2. If we choose f to be meromorphic and g to be entire of finite order or of non zero lower order satisfying $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$, $\lambda_f^{L^*} > 0$ and $\sum\limits_{a\neq\infty}\Theta\left(a; g\right) = 2$, then Theorem 1 remains true with $P_0\left[f\right]$ replaced by $P_0\left[g\right]$ in the denominator.

Remark 3. By Lemma 6 the conclusion of Remark 2 can also drawn under the hypothesis $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ instead of $\sum_{a \neq \infty} \Theta(a; g) = 2$.

In the line of Theorem 1 and with the help of Lemma 8 we may state the following theorem without proof :

Theorem 2. Let f be transcendental meromorphic with finite order or non zero lower order and g be entire satisfying the following conditions:
(i) $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and (ii) ${}_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Then for any $A > 0$

$$\limsup_{r \to \infty} \frac{\log^{[2]} T\left(\exp\left(r^A\right), f \circ g\right)}{\log T\left(\exp\left(r^\mu\right), M[f]\right) + K(r, g; L)} = \infty ,$$

where $0 < \mu < \rho_g$ and $K(r, g; L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \hfill \text{as } r \to \infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise .} \end{cases}$

Remark 4. If we choose f to be meromorphic and g to be transcendental entire of finite order or of non zero lower order satisfying $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$, $\lambda_f^{L^*} > 0$ and ${}_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$, then Theorem 2 remains true with $M[f]$ replaced by $M[g]$ in the denominator.

Theorem 3. Let f be a meromorphic function with finite order or non zero lower order and g be an entire function such that $0 < \rho_g^{L^*} < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Then

$$\lim_{r \to \infty} \frac{\log\left\{T(r, f \circ g) \log M(r, g)\right\}}{T(r, P_0[f]) \cdot K(r, g; L)} = 0 ,$$

where $K(r, g; L) = \begin{cases} 1 \text{ if } L(M(r, g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\} \text{ as } r \to \infty \\ \hfill \text{and for some } \alpha < \lambda_f^{L^*} \\ L(M(r, g)) \text{ otherwise.} \end{cases}$

Proof. In view of Lemma 1 we have for all sufficiently large values of r that

$$T(r, f \circ g) \log M(r, g) \leq \{1 + o(1)\} T(r, g) T(M(r, g), f)$$

$$\begin{aligned} i.e., \ \log\{T(r, f \circ g) \log M(r, g)\} &\leq \log\{1 + o(1)\} + \log T(r, g) \\ &\quad + \log T(M(r, g), f) \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log\{T(r, f\circ g)\log M(r,g)\} &\leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)\log\left[re^{L(r)}\right] \\ &\quad + \left(\rho_f^{L^*}+\varepsilon\right)\left[\log M(r,g)\,e^{L(M(r,g))}\right] \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log\{T(r, f\circ g)\log M(r,g)\} &\leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)] \\ &\quad + \left(\rho_f^{L^*}+\varepsilon\right)[\log M(r,g) + L(M(r,g))] \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log\{T(r, f\circ g)\log M(r,g)\} &\leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)] \\ &\quad + \left(\rho_f^{L^*}+\varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M(r,g))\right]. \end{aligned} \tag{6}$$

Also in view of Lemma 6 we obtain for all sufficiently large values of r that

$$\begin{aligned} \log T(r, P_0[f]) &\geq \left(\lambda_{P_0[f]}^{L^*}-\varepsilon\right)\log\left[re^{L(r)}\right] \\ i.e., \;\; \log T(r, P_0[f]) &\geq \left(\lambda_f^{L^*}-\varepsilon\right)\log\left[re^{L(r)}\right] \\ i.e., \;\; T(r, P_0[f]) &\geq \left[re^{L(r)}\right]^{\left(\lambda_f^{L^*}-\varepsilon\right)} . \end{aligned} \tag{7}$$

Now from (6) and (7) we get for all sufficiently large values of r that

$$\begin{aligned} \frac{\log\{T(r, f\circ g)\log M(r,g)\}}{T(r, P_0[f])} &\leq \frac{o(1) + \left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)]}{T(r, P_0[f])} \\ &\quad + \frac{\left(\rho_f^{L^*}+\varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M(r,g))\right]}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}} . \end{aligned} \tag{8}$$

Since $\rho_g^{L^*} < \lambda_f^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_g^{L^*} + \varepsilon < \lambda_f^{L^*} - \varepsilon . \tag{9}$$

Case I. Let $L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha < \lambda_f^{L^*}$.
As $\alpha < \lambda_f^{L^*}$ we can choose $\varepsilon\,(>0)$ such that

$$\alpha < \lambda_f^{L^*} - \varepsilon . \tag{10}$$

Since $L\left(M\left(r,g\right)\right)=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ we obtain on using (10) that

$$\frac{L\left(M\left(r,g\right)\right)}{r^{\alpha}e^{\alpha L(r)}}\to 0 \text{ as } r\to\infty$$

$$i.e., \ \frac{L\left(M\left(r,g\right)\right)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{L^*}-\varepsilon\right)}}\to 0 \text{ as } r\to\infty \ . \tag{11}$$

Now in view of (8), (9) and (11) we get that

$$\lim_{r\to\infty}\frac{\log\left\{T\left(r,f\circ g\right)\log M\left(r,g\right)\right\}}{T\left(r,P_0\left[f\right]\right)}=0 \ . \tag{12}$$

Case II. If $L\left(M\left(r,g\right)\right)\neq o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha<\lambda_f^{L^*}$ then we get from (8) that for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log\left\{T\left(r,f\circ g\right)\log M\left(r,g\right)\right\}}{T\left(r,P_0\left[f\right]\right)L\left(M\left(r,g\right)\right)} &\leq \frac{o\left(1\right)+\left(\rho_g^{L^*}+\varepsilon\right)\left[\log\left\{re^{L(r)}\right\}\right]}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L\left(M\left(r,g\right)\right)}\\ &+\frac{\left(\rho_f^{L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L\left(M\left(r,g\right)\right)}\\ &+\frac{1}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L\left(M\left(r,g\right)\right)} \ .\end{aligned} \tag{13}$$

Now using (9) it follows from (13) that

$$\lim_{r\to\infty}\frac{\log\left\{T\left(r,f\circ g\right)\log M\left(r,g\right)\right\}}{T\left(r,P_0\left[f\right]\right)L\left(M\left(r,g\right)\right)}=0 \ . \tag{14}$$

Combining (12) and (14) we obtain that

$$\lim_{r\to\infty}\frac{\log\left\{T\left(r,f\circ g\right)\log M\left(r,g\right)\right\}}{T\left(r,P_0\left[f\right]\right)\cdot K\left(r,g;L\right)}=0 \ ,$$

where $K\left(r,g;L\right)=\begin{cases}1 \text{ if } L\left(M\left(r,g\right)\right)=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad\qquad\qquad \text{and for some } \alpha<\lambda_f^{L^*}\\ L\left(M\left(r,g\right)\right) \text{ otherwise.}\end{cases}$

Thus the theorem is established.

Remark 5. In view of Lemma 5 one can easily verify that the conclusion of Theorem 3 can also be deduced if we replace " $\Theta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta\left(a;f\right)=1$ " by $\sum\limits_{a\neq\infty}\Theta\left(a;f\right)=2$.

Theorem 4. Let f be a transcendental meromorphic function with finite order or non zero lower order and g be an entire function such that $0 < \rho_g^{L^*} < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and ${}_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$. Then

$$\lim_{r\to\infty} \frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,M[f])\cdot K(r,g;L)} = 0 ,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < \lambda_f^{L^*} \\ L(M(r,g)) \text{ otherwise.} \end{cases}$

The proof of the above theorem can be established in the line of Theorem 3 and with the help of Lemma 8 and therefore is omitted.

Theorem 5. Let f be meromorphic and g be entire with finite order or of non zero lower order and $\sum\limits_{a\neq\infty}\Theta(a;g) = 2$. Also Let $0 < \rho_g^{L^*} < \rho_f^{L^*} < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,P_0[g])\cdot K(r,g;L)} = 0 ,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < \rho_f^{L^*} \\ L(M(r,g)) \text{ otherwise.} \end{cases}$

The proof is omitted because it can be carried out in the line of Theorem 3.

Remark 6. By Lemma 6 the conclusion of Theorem 5 can also be drawn under the hypothesis $\Theta(\infty;g) = \sum\limits_{a\neq\infty}\delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum\limits_{a\neq\infty}\delta(a;g) = 1$ instead of $\sum\limits_{a\neq\infty}\Theta(a;g) = 2$.

In the line of Theorem 5 one may state the following theorem without proof :

Theorem 6. Let f be a meromorphic function and g be a transcendental entire function with finite order or of non zero lower order and ${}_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Also let $0 < \rho_g^{L^*} < \rho_f^{L^*} < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,M[g])\cdot K(r,g;L)} = 0 ,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } L(M(r,g)) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \alpha < \rho_f^{L^*} \\ L(M(r,g)) \text{ otherwise.} \end{cases}$

Theorem 7. Let f be a meromorphic function with finite order or non zero lower order and $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Also let g be entire.If $\rho_f^{L^*} < \infty$ and $\lambda_{f \circ g}^{L^*} = \infty$ then

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, P_0[f])} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log T(r, f \circ g) \leq \beta \log T(r, P_0[f]). \tag{15}$$

Again from the definition of $\rho_{P_0[f]}^{L^*}$ it follows that for all sufficiently large values of r and in view of Lemma 6

$$\begin{aligned} \log T(r, P_0[f]) &\leq \left(\rho_{P_0[f]}^{L^*} + \varepsilon\right) \log \left\{re^{L(r)}\right\} \\ i.e., \ \log T(r, P_0[f]) &\leq \left(\rho_f^{L^*} + \varepsilon\right) \log \left\{re^{L(r)}.\right\} \qquad (16) \end{aligned}$$

Thus from (15) and (16) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T(r, f \circ g) &\leq \beta \left(\rho_f^{L^*} + \varepsilon\right) \log \left\{re^{L(r)}\right\} \\ i.e., \ \frac{\log T(r, f \circ g)}{\log \left(re^{L(r)}\right)} &\leq \frac{\beta \left(\rho_f^{L^*} + \varepsilon\right) \log \left\{re^{L(r)}\right\}}{\log \left\{re^{L(r)}\right\}} \\ i.e., \ \liminf_{r \to \infty} \frac{\log T(r, f \circ g)}{\log \left(re^{L(r)}\right)} &= \lambda_{f \circ g}^{L^*} < \infty. \end{aligned}$$

This is a contradiction.
This proves the theorem.

Remark 7. Theorem 7 is also valid with "limit superior" instead of "limit" if $\lambda_{f \circ g}^{L^*} = \infty$ is replaced by $\rho_{f \circ g}^{L^*} = \infty$ and the other conditions remaining the same.

Corollary 1. Under the assumptions of Theorem 7 or Remark 7,

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[f])} = \infty.$$

Proof. By Theorem 7 or Remark 7 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log T(r, f \circ g) &> K \log T(r, P_0[f]) \\ i.e.,\ T(r, f \circ g) &> \log \{T(r, P_0[f])\}^K , \end{aligned}$$

from which the corollary follows.

Remark 8. The condition $\lambda_{f \circ g}^{L^*} = \infty$ is necessary in Theorem 7 and Corollary 1 which is evident from the following example :

Example 1. Let $f = \exp z,\ g = z$ and $L(r) = \frac{1}{p} \exp\left(\frac{1}{r}\right)$ where p is any positive real number.
Also let $s = 1,\ A_1 = 1$ and

$$\begin{aligned} n_{i1} &= 1 \text{ for } i = 1 \\ &= 0 \text{ for i} \neq 1. \end{aligned}$$

Then

$$P_0[f] = \exp z.$$

Also

$$\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1,\ \rho_f^{L^*} = 1 < \infty \text{ and } \lambda_{f \circ g}^{L^*} = 1 < \infty.$$

Now

$$T(r, f \circ g) = T(r, \exp z) = \frac{r}{\pi}$$

and

$$T(r, P_0[f]) = T(r, \exp z) = \frac{r}{\pi}.$$

Therefore

$$\begin{aligned} \lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, P_0[f])} &= \lim_{r \to \infty} \frac{\log r + O(1)}{\log r + O(1)} = 1 \text{ and} \\ \lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[f])} &= \lim_{r \to \infty} \frac{\left(\frac{r}{\pi}\right)}{\left(\frac{r}{\pi}\right)} = 1. \end{aligned}$$

Remark 9. Considering

$$\begin{aligned} f &= \exp z,\ g = z,\ A = 1, L(r) = \frac{1}{p} \exp\left(\frac{1}{r}\right) \\ &\quad \text{where } p \text{ is any positive real number;} \\ s &= 1, A_1 = 1 \text{ and} \\ n_{i1} &= 1 \text{ for } i = 1 \\ &= 0 \text{ for i} \neq 1. \end{aligned}$$

one can also verify that the condition $\rho_{f \circ g}^{L^*} = \infty$ in Remark 7 and Corollary 1 is essential.

Remark 10. The conclusion of Theorem 7, Remark 7 and Corollary 1 can also drawn under the hypothesis $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\sum_{a \neq \infty} \Theta(a; f) = 2$ instead of $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$.

In the line of Theorem 17 the following theorem may be deduced:

Theorem 8. Let f be a transcendental meromorphic function with finite order or non zero lower order and $_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Also let g be entire.If $\rho_f^{L^*} < \infty$ and $\lambda_{f \circ g}^{L^*} = \infty$ then

$$\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r, M[f])} = \infty.$$

Remark 11. Theorem 8 is also valid with "limit superior" instead of "limit" if $\lambda_{f \circ g}^{L^*} = \infty$ is replaced by $\rho_{f \circ g}^{L^*} = \infty$ and the other conditions remaining the same.

Corollary 2. Under the assumptions of Theorem 8 or Remark 11,

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, M[f])} = \infty.$$

The proof is omitted because it can be carried out in the line of Corollary 1.

Theorem 9. Let f be a meromorphic function with finite order or non zero lower order and $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$.Also let g be an entire function and $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \sigma_g^{L^*} < \infty$. If $L(M(r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then

$$\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T\left(\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0[f]\right)} \leq \frac{\rho_f^{L^*} \cdot \sigma_g^{L^*}}{\lambda_f^{L^*}}.$$

Proof. Since $T(r, g) \leq \log^+ M(r, g)$ and by Lemma 1 we get for all sufficiently large values of r that

$$\log T(r, f \circ g) \leq \log\{1 + o(1)\} + \log T(M(r, g), f)$$
$$i.e., \quad \log T(r, f \circ g) \leq o(1) + \log T(M(r, g), f)$$

$$i.e., \ \frac{\log T\left(r, f\circ g\right)}{\log T\left(\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right)}$$
$$\leq \frac{o\left(1\right)+\log T\left(M\left(r,g\right),f\right)}{\log T\left(\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right)} = \frac{o\left(1\right)+\log T\left(M\left(r,g\right),f\right)}{\log\left\{M\left(r,g\right)e^{L(M(r,g))}\right\}}\cdot$$
$$\frac{\log M\left(r,g\right)+L\left(M\left(r,g\right)\right)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}}}\cdot\frac{\log\left\{\exp\left(re^{L(r)}\right)^{\rho_g^{L^*}}\right\}}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right]} \tag{17}$$

$$i.e., \ \limsup_{r\to\infty}\frac{\log T\left(r, f\circ g\right)}{\log T\left(\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right)}$$
$$\leq \limsup_{r\to\infty}\frac{\log T\left(M\left(r,g\right),f\right)}{\log\left\{M\left(r,g\right)e^{L(M(r,g))}\right\}}\cdot\limsup_{r\to\infty}\frac{\log M\left(r,g\right)+L\left(M\left(r,g\right)\right)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}}}\cdot$$
$$\limsup_{r\to\infty}\frac{\log\left\{\exp\left(re^{L(r)}\right)^{\rho_g^{L^*}}\right\}}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right]} \ . \tag{18}$$

As $\alpha < \rho_g^{L^*}$ we can choose $\varepsilon\left(>0\right)$ in such a way that $\alpha < \rho_g^{L^*}-\varepsilon$ and since $L\left(M\left(r,g\right)\right) = o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{L\left(M\left(r,g\right)\right)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}-\varepsilon}} = 0 \ . \tag{19}$$

Now from (18) and (19) and in view of Lemma 6 it follows that

$$\limsup_{r\to\infty}\frac{\log T\left(r, f\circ g\right)}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right]} \leq \rho_f^{L^*}\cdot\sigma_g^{L^*}\cdot\frac{1}{\lambda_{P_0[f]}^{L^*}}$$
$$i.e., \ \limsup_{r\to\infty}\frac{\log T\left(r, f\circ g\right)}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0\left[f\right]\right]} \leq \rho_f^{L^*}\cdot\sigma_g^{L^*}\cdot\frac{1}{\lambda_f^{L^*}} \ .$$

Thus the theorem is established.

Remark 12. By Lemma 5 one can verify that the Theorem 9 is also valid if we take $\sum\limits_{a\neq\infty}\Theta\left(a;f\right) = 2$ instead of " $\Theta\left(\infty;f\right) = \sum\limits_{a\neq\infty}\delta_p\left(a;f\right) = 1$ or $\delta\left(\infty;f\right) = \sum\limits_{a\neq\infty}\delta\left(a;f\right) = 1$ " and the other conditions are remaining the same.

In the line of Theorem 9 the following theorem can be proved :

Theorem 10. Let f be meromorphic and g be entire of finite order or of non zero lower order such that $\lambda_g^{L^*} > 0, 0 < \rho_f^{L^*} < \infty$, $0 < \sigma_g^{L^*} < \infty$ and $\sum_{a \neq \infty} \Theta(a; g) = 2$. If $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, P_0[g]\right]} \leq \frac{\rho_f^{L^*} \cdot \sigma_g^{L^*}}{\lambda_g^{L^*}} .$$

The proof is omitted.

Remark 11. The conclusion of Theorem 10 can also be drawn under the hypothesis " $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ " instead of $\sum_{a \neq \infty} \Theta(a; g) = 2$.

Theorem 11. Let f be a transcendental meromorphic function with finite order or non zero lower order and ${}_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Also let g be an entire function and $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \sigma_g^{L^*} < \infty$. If $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, M[f]\right]} \leq \frac{\rho_f^{L^*} \cdot \sigma_g^{L^*}}{\lambda_f^{L^*}} .$$

Theorem 12. Let f be meromorphic and g be transcendental entire of finite order or of non zero lower order such that $\lambda_g^{L^*} > 0$, $0 < \rho_f^{L^*} < \infty$, $0 < \sigma_g^{L^*} < \infty$ and ${}_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. If $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some positive $\alpha < \rho_g^{L^*}$, then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{\log T\left[\exp\left\{re^{L(r)}\right\}^{\rho_g^{L^*}}, M[g]\right]} \leq \frac{\rho_f^{L^*} \cdot \sigma_g^{L^*}}{\lambda_g^{L^*}} .$$

The proof of the above two theorems can be established in the line of Theorem 9 and Theorem 10 respectively and with the help of Lemma 8 and therefore is omitted.

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