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QUASI-REDUCED RINGS

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Abstract. Let R be an arbitrary ring with identity. In this paper, we introduce quasi-reduced rings as a generalization of reduced rings and investigate their properties. The ring R is called *quasi-reduced* if for any $a,\ b \in R$, $ab = 0$ implies $(aR) \cap (Rb)$ is contained in the center of R. We prove that some results of reduced rings can be extended to quasi-reduced rings for this general settings.

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1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Recently a generalization of semicommutative rings is given in [1]. A ring R is called *central semicommutative* if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$. A ring R is called *right (left) principally quasi-Baer* [3] if the right (left) annihilator of a principal right (left) ideal of R is generated by an idempotent. Finally, a ring R is called *right (left) principally projective* if the right (left) annihilator of an element of R is generated by an idempotent [2].

In this paper, we introduce quasi-reduced rings. This class of rings generalizes reduced rings. Since every reduced ring is quasi-reduced, we investigate sufficient conditions for quasi-reduced rings to be reduced. We show that some results of reduced rings can be extended to quasi-reduced rings for this general settings. We give some examples to show that all quasi-reduced rings need not be reduced. Among others we prove that quasi-reduced rings are abelian and there exists an abelian ring which is not quasi-reduced. Therefore the class of quasi-reduced rings lies strictly between the classes of reduced rings and abelian rings. It is shown that every quasi-reduced ring is weakly semicommutative, central semicommutative, 2-primal, abelian and so directly finite. We prove that a ring R is quasi-reduced if and only if the Dorroh extension of R is quasi-reduced. We show that Köthe's conjecture holds for the class of quasi-reduced rings.

Throughout this paper, let $\mathbb{Z}$ be the ring of integers, and for a positive integer n, $\mathbb{Z}_n$ and $\mathbb{Z}^{2\times 2}$ denote the ring of integers modulo n and the ring of 2×2 matrices over $\mathbb{Z}$, respectively. We write $R[x]$ and $R[x, x^{-1}]$ for the polynomial ring and the Laurent polynomial ring over R, respectively.

2. Quasi-Reduced Rings

According to Lee and Zhou [8], a ring R is called *reduced* if for any a, $b \in R$, $ab = 0$ implies $(aR) \cap (Rb) = 0$. In the context, a reduced ring is known as to be a ring with no nonzero nilpotent elements. Actually, for a ring R, for any a, $b \in R$, $ab = 0$ implies $(aR) \cap (Rb) = 0$ if and only if R does not contain nonzero nilpotents if and only if for any $a \in R$, $a^2 = 0$ implies $a = 0$. However if the ring R does not have an identity these reduced concepts are not equivalent as the following example shows.

Example 2.1 Consider the subring R (without unit) of the ring of 2×2 matrices over $\mathbb{Z}_2$ of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$ (see [6, Ex. 1.10]). The ring R is noncommutative. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a nonzero nilpotent element of R. Let $\begin{bmatrix} a & b \\ a & b \end{bmatrix}, \begin{bmatrix} c & d \\ c & d \end{bmatrix} \in R$ with $\begin{bmatrix} a & b \\ a & b \end{bmatrix}\begin{bmatrix} c & d \\ c & d \end{bmatrix} = 0$. Then $(a + b)c = 0$ and $(a + b)d = 0$. Two cases arise here. $a + b = 0$ or $a + b \neq 0$. If $a + b = 0$, then $a = b = 0$ or $a = b = 1$. These cases imply $\begin{bmatrix} a & b \\ a & b \end{bmatrix} R \cap R \begin{bmatrix} c & d \\ c & d \end{bmatrix} = 0$. Assume $a + b \neq 0$. Then $c = d = 0$. Hence $\begin{bmatrix} a & b \\ a & b \end{bmatrix} R \cap R \begin{bmatrix} c & d \\ c & d \end{bmatrix} = 0$.

Now, we give the definition of quasi-reduced rings.

Definition 2.2 The ring R is called *quasi-reduced* if for any a, $b \in R$, $ab = 0$ implies $(aR) \cap (Rb)$ is contained in the center of R.

Commutative rings and reduced rings are quasi-reduced. One may suspect that quasi-reduced rings are reduced. But the following example erases the possibility.

Example 2.3 Consider the ring $R = \mathbb{Z}[x]/(x^2)$. Then R is a commutative ring and so R is quasi-reduced. If $a = x + (x^2) \in R$, then $a^2 = 0$. Therefore R is not a reduced ring.

In general, one may prove that every subring of a quasi-reduced ring is quasi-reduced and any finite direct sum of quasi-reduced rings is quasi-reduced.

Lemma 2.4 *Let R be a ring.*

(1) *If R is a quasi-reduced ring, then for $a \in R$, $a^2 = 0$ implies a is central. The converse holds if R is a semiprime ring.*

(2) *If R is a quasi-reduced ring, then it is abelian, i.e. any idempotent of R is central.*

Proof. (1) Assume that R is a quasi-reduced ring. Let $a \in R$ with $a^2 = 0$. Then $(aR) \cap (Ra)$ is central. Since $a \in (aR) \cap (Ra)$, a is central. Conversely, assume that R is a semiprime ring and whenever $a^2 = 0$ implies a is central. Let a, $b \in R$ with $ab = 0$. Then ba, bra and arb are central for $r \in R$. Let $at = sb \in (aR) \cap (Rb)$ where t, $s \in R$. Then $ata = sba$ and $(ata)^2 = (sba)^2 = sbasba = sbabas = 0$. Hence $ata = sba$ is central and so $ata^2 = sbaa = asba = abas = 0$. On the other hand, $atat = tata$ and $atata = tata^2 = 0$. It follows that $(at)^3 = 0$. Hence $(at)^2$ is central. Then $((at)^2R)^2 = 0$. Hence $(at)^2 = 0$. By assumption, at is central. Therefore R is quasi-reduced.

(2) Let e be an idempotent in R. By (1), $(er - ere)^2 = 0$ implies $er - ere$ central for all $r \in R$. Commuting $er - ere$ by e we have $er = ere$. Similarly, $(re - ere)^2 = 0$ implies $re = ere$ for all $r \in R$. Hence e is central. □

Corollary 2.5 *Let R be a quasi-reduced ring and $a, b \in R$ with $ab = 0$. Then the element ba of R is central.*

The following example shows that the converse of Lemma 2.4(2) need not be true in general.

Example 2.6 Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{Z}^{2\times 2} \mid a \equiv d \ (mod\ 2), b \equiv c \equiv 0 \ (mod\ 2) \right\}.$$

Since only idempotents of R are identity and zero matrices, R is abelian. If $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in R$, then $A^2 = 0$ but A is not central. Therefore R is not quasi-reduced by Lemma 2.4(1).

Theorem 2.7 *Let R be a ring. If every nilpotent element of R is central, then it is quasi-reduced. The converse holds if R is semiprime.*

Proof. Let a, $b \in R$ with $ab = 0$. Then $(ba)^2 = 0$ implies ba is central in R. Let $x = aa_1 = b_1 b \in (aR) \cap (Rb)$, where $a_1, b_1 \in R$. Then $x^3 = b_1 bb_1 bb_1 b = b_1 bb_1 baa_1 = b_1 babb_1 a_1 = 0$. By hypothesis, x is central. Hence R is quasi-reduced. For the converse assume that R is semiprime and quasi-reduced. Let $a \in R$ with $a^n = 0$ for some positive integer n. By Lemma 2.4, we may suppose $n \geq 3$. Then $(a^{n-1})^2 = 0$ and so a^{n-1} is central. Hence $a^{n-1}Ra^{n-1} = 0$. Being R semiprime we have $a^{n-1} = 0$. So in the same way we may induce to the case $a^2 = 0$. By Lemma 2.4(1), a is central. □

In Theorem 2.7 being R a semiprime ring is not superfluous. However we should find an example to establish this claim for rings with identity. But we are not able to find an example for rings with identity. For rings with no identity the following example establishes the claim.

Example 2.8 Consider the ring R as in Example 2.1 and let $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then R is quasi-reduced. On the other hand, it is not semiprime since Ra is a nonzero nilpotent left ideal. Also, the element a is nilpotent which is not central in R.

Recall that a ring R is said to have *nilpotency index* if there is a positive integer n such that $a^n = 0$ for all nilpotent elements $a \in R$. Then we have the following result by Lemma 2.4(1) and Theorem 2.7.

Corollary 2.9 *Let R be a ring with the nilpotency index 2. Then the following are equivalent.*

(1) *R is quasi-reduced.*

(2) *Every nilpotent element of R is central.*

Proposition 2.10 *If R is a reduced ring, then R is quasi-reduced. The converse holds if R satisfies any of the following conditions.*

(1) *R is a semiprime ring.*

(2) *R is a right (left) principally projective ring.*

(3) *R is a right (left) principally quasi-Baer ring.*

Proof. First statement is clear. Conversely, assume that R is a quasi-reduced ring and $a \in R$ with $a^2 = 0$. By Lemma 2.4(1), a is central. Now consider the following cases.

(1) Let R be a semiprime ring. Since a is central, $axa = 0$ for all $x \in R$. It follows that $a = 0$. Therefore R is reduced.

(2) Assume R is a right principally projective ring. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Thus $a = ea = ae = 0$, and so R is reduced. A similar proof may be given for left principally projective rings.

(3) Same as the proof of (2). □

Corollary 2.11 *If R is a quasi-reduced ring, then the following conditions are equivalent.*

(1) *R is a right principally projective ring.*

(2) *R is a left principally projective ring.*

(3) *R is a right principally quasi-Baer ring.*

(4) *R is a left principally quasi-Baer ring.*

Proof. It follows from Proposition 2.10 since in either case R is reduced. □

Note that the homomorphic image of a quasi-reduced ring need not be quasi-reduced. Consider the following example.

Example 2.12 Let D be a division ring, $R = D < x, y >$ and $I =< x^2 >$ where $xy \neq yx$. Since R is a domain, it is quasi-reduced. On the other hand, $\overline{x}^2 = \overline{0}$ but $\overline{x}$ is not a central element of R/I. Then R/I is not quasi-reduced by Lemma 2.4(1).

Proposition 2.13 *Let R be a ring. Then the following are equivalent.*

(1) *R is a domain.*

(2) *R is prime and reduced.*

(3) *R is prime and quasi-reduced.*

Proof. (1) ⇒ (2) ⇒ (3) Clear.

(3) ⇒ (1) Let $a, b \in R$ with $ab = 0$. Then $abr = 0$ for all $r \in R$, and by Corollary 2.5, bRa is contained in the center of R. Hence we have $(asb)R(asb) = 0$ for any $s \in R$. Since R is prime, $asb = 0$ for any $s \in R$ and so $aRb = 0$. This implies that $a = 0$ or $b = 0$. Therefore R is a domain. □

Let $P(R)$ denote the prime radical and $N(R)$ the set of all nilpotent elements of the ring R. The ring R is called 2-*primal* if $P(R) = N(R)$ (see namely [4] and [5]). Lemma 2.14 is well known and easy to prove.

Lemma 2.14 *Let R be a ring. Then we have the following.*

(1) *If R is 2-primal and semiprime, then it is reduced.*

(2) *If R is semicommutative and semiprime, then it is reduced.*

In [11, Theorem 1.5] it is proved that every semicommutative ring is 2-primal. In this direction we prove the next result.

Theorem 2.15 *Every quasi-reduced ring is 2-primal. The converse holds for semiprime rings.*

Proof. It is well known that $P(R) \leq N(R)$. Let $a \in R$ and $a^n = 0$ for some integer $n \geq 2$. For any $r_1 \in R$, $ar_1a^{n-1} \in (aR) \cap (Ra^{n-1})$ is central. Commuting ar_1a^{n-1} with ar_2 for any $r_2 \in R$, we have

$$ar_2ar_1a^{n-1} = ar_1a^{n-1}ar_2 = 0.$$

By hypothesis, for any $s_1 \in R$, $ar_2ar_1as_1a^{n-2}$ is central. Commuting $ar_2ar_1as_1a^{n-2}$ with ar_3 for any $r_3 \in R$, we have

$$ar_3ar_2ar_1as_1a^{n-2} = ar_2ar_1as_1a^{n-2}ar_3 = 0.$$

By hypothesis, for any $s_2 \in R$, $ar_3ar_2ar_1as_1as_2a^{n-3}$ is central. Commuting it with ar_4 for any $r_4 \in R$, we have

$$ar_4ar_3ar_2ar_1as_1as_2a^{n-3} = ar_3ar_2ar_1as_1as_2a^{n-3}ar_4 = 0.$$

By hypothesis, for any $s_3 \in R$, $ar_4ar_3ar_2ar_1as_1as_2as_3a^{n-4}$ is central. Commuting it by ar_5 for any $r_5 \in R$, we have

$$ar_5ar_4ar_3ar_2ar_1as_1as_2as_3a^{n-4} = ar_4ar_3ar_2ar_1as_1as_2as_3a^{n-4}ar_5 = 0.$$

There exists a positive integer t depending on n such that $ax_1ax_2ax_3...ax_ta = 0$ for all $x_i \in R$, where $(i = 1, 2, 3, ..., t)$. Then for any prime ideal P, we have $aR(ax_2ax_3...ax_ta) \leq P$. So $a \in P$ or $ax_2ax_3...ax_ta \in P$ for all $x_2, ..., x_t$. If $aRax_3...ax_ta \leq P$, then $a \in R$ or $ax_3...ax_ta \in P$ for all $x_3, ..., x_t$. Continuing in this way we reach $a \in P$ for all prime ideals P. Thus $a \in P(R)$ and so $N(R) = P(R)$. Conversely, let R be a semiprime and 2-primal ring. Then R is reduced by Lemma 2.14 and so quasi-reduced. □

Corollary 2.16 *Let R be a quasi-reduced ring. Then the ring $R/P(R)$ is reduced.*

Proof. Clear by Theorem 2.15 since $P(R)$ consists of all nilpotent elements. □

Theorem 2.17 *Every quasi-reduced ring is central semicommutative. The converse holds for semiprime rings.*

Proof. Let R be a quasi-reduced ring and $a, b \in R$ with $ab = 0$. Then $(aR) \cap (Rb)$ is contained in the center of R. Since $aRb \leq (aR) \cap (Rb)$, R is central semicommutative. The converse is clear from the fact that every semiprime central semicommutative ring is reduced by Lemma 2.14. □

The *Köthe's conjecture* states that if R has no nonzero nil ideals, then R has no nonzero nil one-sided ideals (see for detail [12]).

Corollary 2.18 *Köthe's conjecture holds for a quasi-reduced ring.*

The converse statement of Theorem 2.17 need not hold in general.

Example 2.19 Let F be a field and consider the subring
$R = \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & b & e \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} \mid a, b, c, d, e \in F \right\}$ of the ring of all 4×4 matrices over F. We show that R is central semicommutative but not quasi-reduced. Let $A, B \in R$ with $AB = 0$. Then we have the following cases:

(1) $A = 0$ or $B = 0$ or

(2) $A = \begin{bmatrix} 0 & b & c & d \\ 0 & 0 & b & e \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & c' & d' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or

(3) $A = \begin{bmatrix} 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b' & c' & d' \\ 0 & 0 & b' & e' \\ 0 & 0 & 0 & b' \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or

(4) $A = \begin{bmatrix} 0 & 0 & c & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & c' & d' \\ 0 & 0 & 0 & e' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

In either case we have $ARB = 0$. Hence R is semicommutative.

For $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $AR \cap RB = RB$ which is not central. Hence R is not quasi-reduced.

Recall that a ring R is called *weakly semicommutative* [9] if for any $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$.

Proposition 2.20 *Let R be a quasi-reduced ring. Then R is weakly semicommutative.*

Proof. Let $a, b \in R$ with $ab = 0$. Since R is quasi-reduced, by Corollary 2.5, ba is central in R. Hence for each $r \in R$, $(arb)^2 = arbarb = ar^2bab = 0$. Therefore R is a weakly semicommutative ring. $\square$

The following example shows that there is a weakly semicommutative ring which is not quasi-reduced.

Example 2.21 Let D be a division ring and consider the 2×2 upper triangular matrix ring $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$. In [9], it is proved that R is weakly semicommutative. Now consider the element $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ of R. Then $a^2 = 0$ and it is clear that a is not central. Hence R is not quasi-reduced by Lemma 2.4(1).

The next example shows that for a ring R and an ideal I, if R/I is quasi-reduced, then R need not be quasi-reduced.

Example 2.22 Let F be a field and consider the 2×2 upper triangular matrix ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then R is not quasi-reduced by the preceding example. Now consider the ideal $I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ of R. Then R/I is quasi-reduced because the ring R/I is commutative.

Proposition 2.23 *Let R be a prime ring. If R/I is a quasi-reduced ring with a reduced ideal I, then R is reduced and therefore quasi-reduced.*

Proof. Let R/I be a quasi-reduced ring. By Theorem 2.17, R/I is central semicommutative. Let $a \in R$ with $a^2 = 0$. Then $(a+I)^2 = 0 \in R/I$, and so $(a+I)(R/I)(a+I)$ is contained in the center of R/I. Hence $(a+I)(R/I)(a+I)(R/I)(a+I) = 0$, so we have $aRaRa \subseteq I$. Let $r, s \in R$. Since $(arasa)^2 = 0$ and I is reduced, $arasa = 0$. Then $aRaRa = 0$. By hypothesis $a = 0$, therefore R is reduced. □

Recall that a ring R is called *directly finite* whenever $a, b \in R$, $ab = 1$ implies $ba = 1$. Then we have the following.

Corollary 2.24 *If R is a quasi-reduced ring, then R is directly finite.*

Proof. By Lemma 2.4(2), R is an abelian ring and so directly finite. □

Recall that a ring R is *von Neumann regular* if for each $a \in R$, there exists $b \in R$ with $aba = a$, while R is *strongly regular* if for each $a \in R$, there exists $b \in R$ with $a^2b = a$. Every strongly regular ring is von Neumann regular.

Theorem 2.25 *The following are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *Every right R-module is flat and R is quasi-reduced.*

(3) *Every cyclic right R-module is flat and R is quasi-reduced.*

(4) *R is regular and quasi-reduced.*

(5) *R is regular and reduced.*

(6) *R is regular and abelian.*

Proof. (1) $\Rightarrow$ (2) Note that every strongly regular ring is von Neumann regular, by Harada's theorem, every module is flat. Being R strongly regular, it does not contain nonzero nilpotent elements. For a, $b \in R$ assume that $ab = 0$. Then $ba = 0$. Let $ar = tb \in (aR) \cap (Rb)$. Multiplying $ar = tb$ by ar we have $(ar)^2 = 0$. So $ar = 0$. Hence $(aR) \cap (Rb) = 0$.

(2) $\Rightarrow$ (3) Obvious. (3) $\Rightarrow$ (4) Clear by [7, Theorem 4.21].

(4) $\Rightarrow$ (1) Let $a \in R$. Then $a = aba$ for some $b \in R$ and so ab is an idempotent. By Lemma 2.4, it is central. Hence $a = aba = a^2b$. The rest is clear. □

Let R be a ring and the Dorroh extension $D(R, \mathbb{Z})$ of R is a ring with componentwise addition and multiplication defined by

$$(r_1, n_1)(r_2, n_2) = (r_1r_2 + r_1n_2 + r_2n_1, n_1n_2)$$

where $r_i \in R$, $n_i \in \mathbb{Z}$ for $i = 1, 2$. It is well known that if R is a reduced ring, then its Dorroh extension has the 2-primal condition ([10]). In this direction we prove the following.

Proposition 2.26 *Let R be a quasi-reduced ring. Then $D(R, \mathbb{Z})$ is also quasi-reduced.*

Proof. Let (r_1, n_1), $(r_2, n_2) \in D(R, \mathbb{Z})$ with $(r_1, n_1)(r_2, n_2) = (r_1r_2 + r_1n_2 + r_2n_1, n_1n_2) = 0$. Then $n_1n_2 = 0$ and $r_1r_2 + r_1n_2 + r_2n_1 = 0$. We divide the proof in two cases: $n_1 = 0$ and $n_2 \neq 0$ or $n_1 \neq 0$ and $n_2 = 0$. We prove only for the case $n_1 = 0$ and $n_2 \neq 0$. Then $r_1r_2 + r_1n_2 = 0$. Since R has identity, $r_2 + n_2$ is an element of R and so $r_1(r_2 + n_2) = 0$. By hypothesis $(r_1R) \cap (R(r_2 + n_2))$ is central. Let $(r_1, 0)(a, n) = (b, m)(r_2, n_2) \in ((r_1, 0)D(R, \mathbb{Z})) \cap (D(R, \mathbb{Z})(r_2, n_2))$. Then $m = 0$ and $r_1(a + n) = b(r_2 + n_2) \in (r_1R) \cap (R(r_2 + n_2))$ is central. Hence $((r_1, 0)D(R, \mathbb{Z})) \cap (D(R, \mathbb{Z})(r_2, n_2))$ is central in this case. The proof for the case $n_1 \neq 0$ and $n_2 = 0$ is similar. $\square$

Let S denote a multiplicatively closed subset of R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S. Then

Proposition 2.27 *R is quasi-reduced if and only if so is $S^{-1}R$.*

Proof. Note that $r/s \in S^{-1}R$ is central in $S^{-1}R$ if and only if r is central in R. Assume that R is a quasi-reduced ring and let a/s, $b/t \in S^{-1}R$, where s, $t \in S$, and $(a/s)(b/t) = 0$. Since all elements of S are central regular, $ab = 0$. By assumption $(aR) \cap (Rb)$ is contained in the center of R. Let $(a/s)(a_1/s_1) = (b_1/t_1)(b/t) \in ((a/s)S^{-1}R) \cap (S^{-1}R)(b/t)$. Then $tt_1aa_1 = ss_1b_1b$ and, since t, t_1, s, s_1 are central, $tt_1aa_1 = ss_1b_1b \in (aR) \cap (Rb)$ and it is central. By dividing $tt_1aa_1 = ss_1b_1b$ by tt_1ss_1, $(a/s)(a_1/s_1) = (b_1/t_1)(b/t)$ is a central element of $((a/s)(S^{-1}R)) \cap ((S^{-1}R)(b/t))$. The converse is clear since R may be embedded in $S^{-1}R$ as a subring and quasi-reducedness is preserved under subrings. $\square$

Corollary 2.28 *For any ring R, the polynomial ring $R[x]$ is quasi-reduced if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is quasi-reduced.*

Proof. Let $S = \{1, x, x^2, x^3, x^4, ...\}$. Then S is a multiplicatively closed subset of $R[x]$ consisting of central regular elements. Then the proof follows from Proposition 2.27. $\square$

Let R be a ring and M be an (R, R)-bimodule. Recall that *the trivial extension of R by M* is defined to be ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This ring is isomorphic to the ring $\left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R, m \in M \right\}$ with the usual matrix operations. The trivial extension of R by M need not be a quasi-reduced ring, as the following example shows.

Example 2.29 Let R be a noncommutative ring and r a noncentral element in R. Consider the element $\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ of $T(R, R)$. Then $\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Let $a \in R$ with $ar \neq ra$. Then we have $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. By Lemma 2.4(1), $T(R, R)$ is not quasi-reduced.

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