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MULTIVALENTLY MEROMORPHIC FUNCTIONS ASSOCIATED WITH CONVOLUTION STRUCTURE

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Abstract. In the present investigation we define a new class of meromorphic functions on the punctured unit disk $\mathcal{U}^* := \{z \in C : 0 < |z| < 1\}$ by making use of the convolution structures.Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning the convolution products and subordination results for functions in new class.

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1. Introduction

Let Σ_p^* denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k, \ (p \in N = \{1, 2, 3 \ldots\}), \tag{1}$$

which are meromorphic and p-valent in the punctured unit disc $\mathcal{U}^* = \{z \in C : 0 < |z| < 1\} = \mathcal{U} \setminus \{0\}$.

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which are meromorphic and p-valent in the punctured unit disc $\mathcal{U}^* = \{z \in C : 0 < |z| < 1\} = \mathcal{U} \setminus \{0\}$.

Let $g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_k z^k, \ (b_k \geq 0; p \in N = \{1, 2, 3 \ldots\})$, then the convolution (or Hadamard) product of $f(z)$ is defined as

$$f(z) * g(z) = (f * g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k b_k z^k, \ (a_k, b_k \geq 0; p \in N = \{1, 2, 3 \ldots\}).$$

For the function f, g in the class Σ_p, we define a linear operator $\mathcal{D}_\lambda^n(f*g)(z)$ by the following form

$$\begin{aligned}
\mathcal{D}_\lambda f*g)(z) &= (1+p\lambda)(f*g)(z)+\lambda z(f*g)(z)'),(\lambda\geq 0)\\
\mathcal{D}_\lambda^0(f*g)(z) &= (f*g)(z),\\
\mathcal{D}_\lambda^1(f*g)(z) &= \mathcal{D}_\lambda(f*g)(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}(1+p\lambda+k\lambda)b_k a_k z^k,\\
\mathcal{D}_\lambda^2(f*g)(z) &= \mathcal{D}_\lambda(\mathcal{D}_\lambda'(f*g)(z))=\frac{1}{z^p}+\sum_{k=p}^{\infty}(1+p\lambda+k\lambda)^2 b_k a_k z^k \qquad (3)
\end{aligned}$$

and in general for $n=0,1,2,...$, we can write

$$\mathcal{D}_\lambda^n(f*g)(z)=\frac{1}{z^p}+\sum_{k=p}^{\infty}(1+p\lambda+k\lambda)^n b_k a_k z^k,(n\in N_0=N\cup\{0\};p\in N) \qquad (4)$$

then we can observe easily that for $f\in\Sigma_p$,

$$z\lambda(\mathcal{D}_\lambda^n(f*g)(z))'=\mathcal{D}_\lambda^{n+1}(f*g)(z)-(1+p\lambda)\mathcal{D}_\lambda^n(f*g)(z),(p\in N;n\in N_0). \qquad (5)$$

Now we introduce a new class $\mathcal{M}^n(\lambda,\alpha,\beta)$of meromorphic starlike functions in the parabolic region to study its characteristic properties.

For fixed parameters $\alpha\geq\frac{1}{2+\beta}; 0\leq\beta<1$ the meromorphically p-valent function $f.g\in\Sigma_p(\alpha)$ will be in the class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$ if it satisfies the inequality

$$\Re\left(\frac{-z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha-\alpha\beta\right)\geq\left|\frac{z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha+\alpha\beta\right|,(n\in N_0). \qquad (6)$$

For various choices of g we get different linear operators which have been studied in recent past. We deem it proper to mention below the some of the function classes which emerge from the function class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$ defined above.

Example 1: If $\lambda=1$ and $b_k=1$ or $g(z)=\frac{1}{z^p(1-z)}$, then

$$\begin{gathered}
\mathcal{M}_g^n(1,\alpha,\beta)\equiv\mathcal{MST}(\alpha,\beta)\\
:=\left\{f\in\Sigma_p:\Re\left(\frac{-zf'(z)}{pf(z)}+\alpha-\alpha\beta\right)\geq\left|\frac{zf'(z)}{pf(z)}+\alpha+\alpha\beta\right|\right\} \qquad (7)
\end{gathered}$$

A function in $\mathcal{MST}(\alpha,\beta)$ is called $\beta-$meromorphically starlike of order α, $(0\leq\alpha<1)$ in the parabolic region .

Example 2: If $\lambda=1$ and $g(z)=\frac{1}{z^p(1-z)^{\delta+p}},(\delta>-p)$ and $f\in\Sigma_p$,then $\mathcal{M}_g^n(1,\alpha,\beta)\equiv\mathcal{MST}(\alpha,\beta,\delta)$, if

$$\Re\left(\frac{-z(D^\delta f(z))'}{pD^\delta f(z)}+\alpha-\alpha\beta\right)\geq\left|\frac{z(D^\delta f(z))'}{pD^\delta f(z)}+\alpha+\alpha\beta\right| \qquad (8)$$

where D^{δ} is a well-known Ruscheweyh derivative operator[17].

Example 3: If $\lambda = 1$ and $b_k = \frac{(a)_k}{(c)_k}$ or $g(z) = \sum\limits_{k=p}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}$, then

$$\mathcal{M}_c^a(\alpha, \beta)$$
$$:= \left\{ f \in \Sigma_p : \Re\left(\frac{-z(\mathcal{L}(a,c)f(z))'}{p\mathcal{L}(a,c)f(z)} + \alpha - \alpha\beta \right) \geq \left| \frac{z(\mathcal{L}(a,c)f(z))'}{p\mathcal{L}(a,c)f(z)} + \alpha + \alpha\beta \right| \right\}$$

where $\mathcal{L}(a, c)$ is a well-known Carlson-Shaffer linear operator [3].

Example 4: If $\lambda = 1$ and $b_k = \Gamma_k = \frac{(\alpha_1)_{k+p} \cdots (\alpha_q)_{k+p}}{(\beta_1)_{k+p} \cdots (\beta_s)_{k+p}} \frac{1}{(k+p)!}$ or $g(z) = z^{-p} + \sum\limits_{k=0}^{\infty} \Gamma_k\ a_k\ z^k$, then

$$\mathcal{M}_s^q(\alpha, \beta)$$
$$:= \left\{ f \in \Sigma_p : \Re\ \left(\frac{-z(\mathcal{H}_s^q[\alpha_1]f(z))'}{p\mathcal{H}_s^q[\alpha_1]f(z)} + \alpha - \alpha\beta \right) \geq \left| \frac{z(\mathcal{H}_s^q[\alpha_1]f(z))'}{p\mathcal{H}_s^q[\alpha_1]f(z)} + \alpha + \alpha\beta \right| \right\}$$

where $\mathcal{H}_s^q[\alpha_1]$ is called Dziok-Srivastava operator[6].

Example 5: If $\lambda = 1$ and $b_k = \left(\frac{1}{k+p+1}\right)^{\sigma}$ or $g(z) = z^{-p} + \sum\limits_{k=p}^{\infty} \left(\frac{1}{k+p+1}\right)^{\sigma}\ a_k\ z^k$, then

$$\mathcal{M}^{\sigma}(\alpha, \beta) \equiv \left\{ f \in \Sigma_p : \Re\ \left(\frac{-z(\mathcal{I}^{\sigma} f(z))'}{p\mathcal{I}^{\sigma} f(z)} + \alpha - \alpha\beta \right) \geq \left| \frac{z(\mathcal{I}^{\sigma} f(z))'}{p\mathcal{I}^{\sigma} f(z)} + \alpha + \alpha\beta \right| \right\}$$

where $\mathcal{I}^{\sigma}$ is called Jung-Kim-Srivastava operator.

2. Inclusion properties of the class $\mathcal{M}_g^*(\lambda, \alpha, \beta)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first theorem.

Lemma 1. Let the (non-constant) function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$ if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then

$$z_0 w'(z_0) = \gamma w(z_0), \tag{9}$$

where γ is real number and $\gamma \geq 1$.

The following inclusion property holds true for the class $\mathcal{M}_g^*(\lambda, \alpha, \beta)$.

Theorem 1. For $\lambda > 0, \alpha \geq \frac{1}{2+\beta}; 0 \leq \beta < 1$, then

$$\mathcal{M}_{\lambda}^*(n+1, \alpha, \beta) \subset \mathcal{M}_g^*(\lambda, \alpha, \beta), \quad (n \in N_0). \tag{10}$$

Proof. Let $f(z) \in \mathcal{M}_{\lambda}^*(n+1, \alpha, \beta)$ and suppose that

$$z^{p+1}(\mathcal{D}_{\lambda}^n f * g(z))' = \frac{-p + [p\beta + (\alpha - \beta)(p-n)]w(z)}{1 + \beta w(z)} \tag{11}$$

where the function $w(z)$ is either analytic or meromorphic in $\mathcal{U}$, with $w(0) = 0$. Then, by using

$$z(\mathcal{D}_\lambda^n f(z))' = \mathcal{D}_\lambda^{n+1}(f * g)(z) - (p+1)\mathcal{D}_\lambda^n(f * g)(z) \tag{12}$$

$(f \in \Sigma_p; n \in N_0 = N \cup \{0\}; p \in N)$ and by (11) we have,

$$z^{p+1}(\mathcal{D}_\lambda^n(f * g)(z))' = \frac{-p + [p\beta + (\alpha-\beta)(p-n)]w(z)}{1+\beta w(z)} - \frac{(\alpha-\beta)(p-n)zw'(z)}{(1+\beta w(z))^2}. \tag{13}$$

We claim that $|w(z)| < 1$ for $z \in \mathcal{U}$. Otherwise there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$.

Applying Jack's lemma, we have $z_0 w'(z_0) = \gamma w(z_0), (\gamma \geq 1)$ writing $w(z_0) = e^{i\theta}(0 \leq \theta \leq 2\pi)$ and putting $z = z_0$ in (13),we get

$$\begin{aligned} &\left| \frac{z_0^{p+1}(\mathcal{D}_\lambda^{n+1}(f * g)(z_0)) + p}{\beta z_0^{p+1}(\mathcal{D}_\lambda^{n+1}(f * g)(z_0))' + [p\beta + (\alpha-\beta)(p-n)]} \right|^2 - 1 \\ &= \frac{|(\gamma+1) + \beta e^{i\theta}|^2 - |1 - \beta(\gamma-1)e^{i\theta}|^2}{|1-\beta(\gamma-1)e^{i\theta}|^2} \\ &= \frac{\gamma^2(1-\beta^2) + 2\gamma(1+\beta^2+2\beta cos\theta)}{|1-\beta(\gamma-1)e^{i\theta}|^2} \geq 0 \end{aligned}$$

which obviously contradicts our hypothesis that $f(z) \in \mathcal{M}_\lambda^*(n+1, \alpha, \beta)$. Thus we must have $|w(z)| < 1 (z \in \mathcal{U})$ and so from (13), we conclude that $f(z) \in \mathcal{M}_g^n(\lambda, \alpha, \beta)$ which evidently completes the proof.

Theorem 2. Let μ be a complex number such that $\Re(\mu) > 0$. If $f(z) \in \mathcal{M}_g^n(\lambda, \alpha, \beta)$ then the function $F_\mu(z)$ given by

$$F_\mu(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)dt \quad (f \in \Sigma) \tag{14}$$

is also in the same class $\mathcal{M}_g^n(\lambda, \alpha, \beta)$.

Proof. It is easily seen from definition

$$\mathcal{D}_\lambda^n(f * g)(z) = z^{-p} + \sum_{k=p}^{\infty}(1 + p\lambda + k\lambda)^n b_k a_k z^k, (n \in N_0 = N \cup \{0\}; p \in N) \tag{15}$$

and (14) that

$$z(\mathcal{D}_\lambda^n F_\mu(z))' = \mu \mathcal{D}_\lambda^n(f * g)(z) - (\mu+p)\mathcal{D}_\lambda^n F_\mu(z) \tag{16}$$

thus by setting

$$z^{p+1}(\mathcal{D}_\lambda^n F_\mu(z))' = \frac{-p+[p\beta+(\alpha-\beta)(p-n)]w(z)}{1+\beta w(z)} \tag{17}$$

where $w(z)$ is either analytic or meromorphic in $\mathcal{U}$, with $w(0)=0$. Then, by using (16) and (17) we have

$$z^{p+1}(\mathcal{D}_\lambda^n (f*g)(z))' = \frac{-p+[p\beta+(\alpha-\beta)(p-n)]w(z)}{1+\beta w(z)} - \frac{(\alpha-\beta)(p-n)zw'(z)}{\mu(1+\beta w(z))^2} \tag{18}$$

the remaining part of the proof is similar to that of Theorem 1 and so is omitted.

3. Properties of the class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$

Theorem 3. Let $f\in\Sigma_p$ then f is in the class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=p}^{\infty}(k+p\alpha\beta)(k\lambda+p\lambda+1)^n a_k \le p(1-\alpha\beta) \tag{19}$$

where $\alpha>\frac{1}{2+\beta}; 0\le\beta<1; p\in N, n\in N_0$.
Proof. Suppose that $f\in\mathcal{M}_g^n(\lambda,\alpha,\beta)$ then by the inequality

$$\left|\frac{z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha+\alpha\beta\right| \le \Re\left\{\frac{-z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha-\alpha\beta\right\}, (n\in N_0) \tag{20}$$

that is,

$$\begin{aligned}\Re\left\{\frac{z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha+\alpha\beta\right\} &\le \left|\frac{z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha+\alpha\beta\right| \\ &\le \Re\left\{\frac{-z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}\right\}+\alpha-\alpha\beta\end{aligned}$$

$$\Re\left\{\frac{z(\mathcal{D}_\lambda^n(f*g)(z))'}{p(\mathcal{D}_\lambda^n(f*g)(z))}+\alpha\beta\right\} \quad\le\quad 0.$$

Substituting for $\mathcal{D}_\lambda^n(f*g)(z)$ from (4) and $(\mathcal{D}_\lambda^n(f*g)(z))'$, we get

$$\Re\left(\frac{\frac{-p}{z^p}+\sum_{k=p}^{\infty}k(1+p\lambda+k\lambda)^n b_k a_k z^k}{\frac{p}{z^p}+\sum_{k=p}^{\infty}p(1+p\lambda+k\lambda)^n b_k a_k z^k}+\alpha\beta\right)\le 0.$$

Since $\Re(z) \leq |z|$, we have

$$|-p+\sum_{k=p}^{\infty} k(1+p\lambda+k\lambda)^n b_k a_k z^{k+p} + p\alpha\beta + \alpha\beta \sum_{k=p}^{\infty} p(1+p\lambda+k\lambda)^n b_k a_k z^{k+p}| \leq 0.$$

Hence, by letting $|z| \to 1^-$ we get

$$\sum_{k=p}^{\infty}(k+p\alpha\beta)(1+p\lambda+k\lambda)^n b_k |a_k| \leq p(1-\alpha\beta)$$

which completes the proof.

Theorem 4. If $f \in \mathcal{M}_g^n(\lambda,\alpha,\beta)$, then

$$\left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n b_p} r^{2p}\right) r^{-(p+m)} \leq |f^{(m)}(z)| \leq$$
$$\left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n b_p} r^{2p}\right) r^{-(p+m)}$$
$$(0<|z|=r<1; \alpha \geq \frac{1}{2+\beta}; 0 \leq \beta < 1; p \in N; m \in N, p > m).$$

Proof. Let $f \in \mathcal{M}_g^n(\lambda,\alpha,\beta)$. Then we find from Theorem 3 that

$$p(1+\alpha\beta)(1+2p\lambda)^n b_p \sum_{k=p}^{\infty}|a_k| \leq \sum_{k=p}^{\infty}(k+p\alpha\beta)(1+p\lambda+k\lambda)^n b_k|a_k| \leq p(1-\alpha\beta)$$

which yields, $\sum_{k=p}^{\infty}|a_k| \leq \frac{p(1-\alpha\beta)}{p(1+\alpha\beta)(1+2p\lambda)^n b_p}$. Now by differentiating both sides of $f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k$ with respect to $z, (m-times)$ we have

$$f^{(m)}(z) = \frac{(-1)^m(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty}\frac{(k)!}{(k-m)!}|a_k| z^{(k-m)}$$

$$f^{(m)}(z) = \frac{(-1)^m(p+m-1)!}{(p-1)!} z^{-(p+m)} + \frac{p!}{(p-m)!} z^{(p-m)} \sum_{k=p}^{\infty} a_k$$

$$f^{(m)}(z) \leq \frac{(-1)^m(p+m-1)!}{(p-1)!} z^{-(p+m)} + \frac{p!}{(p-m)!}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n b_p} z^{(p-m)}$$

$$|f^{(m)}(z)| \leq \left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n b_p} r^{2p}\right) r^{-(p+m)}.$$

On the other hand we have

$$|f^m(z)| \geq \left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n b_p} r^{2p}\right) r^{-(p+m)}$$

Hence the proof.

4. The radii of meromorphically starlikeness

Theorem 5. Let the function $f(z)$ defined by (2) be in the class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$ then we have $f(z)$ is meromorphically p-valent starlike of order $\varphi(0 \le \varphi < p)$ in the disc $|z| < r_1$, that is, $\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \varphi, \quad |z| < r_1; 0 \le \varphi < p; p \in N$, where

$$|z| \le \left\{\frac{(k+p\alpha\beta)(p-\mu)(1+k\lambda+p\lambda)^n b_k}{p(k+\mu)(1-\alpha\beta)}\right\}^{\frac{1}{k+P}}. \tag{21}$$

Proof. Let $f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k$ we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| \le \frac{\sum_{k=p}^{\infty}(k+p)a_k|z|^{k+p}}{2(p-\mu)+\sum_{k=p}^{\infty}(k-p+2\mu)a_k|z|^{k+p}}.$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| \le 1, \quad \text{if,} \quad \sum_{k=p}^{\infty}\frac{k+\mu}{p-\mu}|a_k||z|^{k+p} \le 1. \tag{22}$$

Since $f \in \mathcal{M}_g^n(\lambda,\alpha,\beta)$ from Theorem 3 , we have

$$\sum_{k=p}^{\infty}\frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\beta)}b_k|a_k| \le 1. \tag{23}$$

From (22) and (23)

$$\begin{aligned}\frac{k+\mu}{p-\mu}|z|^{k+p} &\le \left\{\frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\beta)}\right\} \\ |z| = r &\le \left\{\frac{(p-\mu)(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\beta)(k+\mu)}\right\}^{\frac{1}{k+p}}\end{aligned}$$

which completes proof .

5. Convolution Properties

For the function

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty}|a_{k,j}|z^k, \quad (j=1,2; p \in N) \tag{24}$$

we denote by $(f_1 * f_2)(z)$ the Hadamard product(Convolution) of the function $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}||a_{k,2}|z^k. \qquad (25)$$

Theorem 6. For the function $f_j(z)$ $(j = 1, 2)$ defined by (24)be in the class $\mathcal{M}_g^n(\lambda, \alpha, \beta)$.Then $(f * g)(z) \in \mathcal{M}_g^n(\lambda, \alpha, \delta)$ where

$$\delta \leq \frac{1}{\alpha}\left(1 - \frac{2(1-\alpha\beta)^2}{(1-\alpha\beta)^2 + (1+\alpha\beta)^2(1+2p\lambda)^n b_k}\right).$$

Proof. Let $f_1(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}|z^k$ and $f_2(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,2}|z^k$ be in the class $\mathcal{M}_g^n(\lambda, \alpha, \beta)$. Then by Theorem 3 , we have

$$\begin{aligned}
\sum_{k=p}^{\infty} \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\beta)} b_k |a_{k,1}| &\leq 1 \\
\sum_{k=p}^{\infty} \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\beta)} b_k |a_{k,2}| &\leq 1.
\end{aligned}$$

Employing the technique used earlier by Schild and Silverman[28],we need to find smallest δ such that

$$\sum_{k=p}^{\infty} \frac{(k+p\alpha\delta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\delta)} b_k |a_{k,1}||a_{k,2}| \leq 1. \qquad (26)$$

By Cauchy-Schwarz inequality, we have

$$\sum_{k=p}^{\infty} \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\beta)} b_k \sqrt{|a_{k,1}||a_{k,2}|} \leq 1 \qquad (27)$$

then

$$\frac{(k+p\alpha\delta)(1+k\lambda+p\lambda)^n |a_{k,1}||a_{k,2}|}{p(1-\alpha\delta)} \leq \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n}{p(1-\alpha\beta)} \sqrt{|a_{k,1}||a_{k,2}|}. \qquad (28)$$

Hence that,

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(k+p\alpha\beta)(1-\alpha\delta)}{(k+p\alpha\delta)(1-\alpha\beta)} \qquad (29)$$

we know that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k} \qquad (30)$$

from (29) and (30), we have

$$\frac{p(1-\alpha\beta)}{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k} \leq \frac{(k+p\alpha\beta)(1-\alpha\delta)}{(k+p\alpha\delta)(1-\alpha\beta)}.$$

It follows that

$$\delta = \frac{1}{\alpha}\left(1 - \frac{p(p+k)(1-\alpha\beta)^2}{p^2(1-\alpha\beta)^2 + (k+p\alpha\beta)^2(1+k\lambda+p\lambda)^n b_k}\right).$$

Now defining a function $\Psi(k)$ by

$$\Psi(k) = \frac{1}{\alpha}\left(1 - \frac{p(p+k)(1-\alpha\beta)^2}{p^2(1-\alpha\beta)^2 + (k+p\alpha\beta)^2(1+k\lambda+p\lambda)^n b_p}\right), (k \geq p),$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\delta = \Psi(p) = \frac{1}{\alpha}\left(1 - \frac{2(1-\alpha\beta)^2}{(1-\alpha\beta)^2 + (1+\alpha\beta)^2(1+2p\lambda)^n b_p}\right).$$

Which completes the proof.

Theorem 7. For the function $f_1(z) \in \mathcal{M}_g^n(\lambda,\alpha,\beta)$ and $f_2(z) \in \mathcal{M}_g^n(\lambda,\alpha,\gamma)$ Then $(f_1 * f_2)(z) \in \mathcal{M}_g^n(\lambda,\alpha,\zeta)$ where

$$\zeta \leq \frac{(1+\alpha\beta)(1+\alpha\gamma)(1+2p\lambda)^n b_p - (1-\alpha\beta)(1-\alpha\gamma)}{\alpha[(1+\alpha\beta)(1+\alpha\gamma)(1+2p\lambda)^n b_p + (1-\alpha\beta)(1-\alpha\gamma)]}.$$

Proof. For the function

$$f_1(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}| z^k \in \mathcal{M}_g^n(\lambda,\alpha,\beta)$$

and

$$f_2(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,2}| z^k \in \mathcal{M}_g^n(\lambda,\alpha,\gamma)$$

we have

$$\sum_{k=p}^{\infty} \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\beta)} |a_{k,1}| \leq 1 \tag{31}$$

$$\sum_{k=p}^{\infty} \frac{(k+p\alpha\gamma)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\gamma)} |a_{k,2}| \leq 1. \tag{32}$$

Since $(f_1 * f_2)(z) \in \mathcal{M}_g^n(\lambda, \alpha, \zeta)$,then by Theorem 3 , we have

$$\sum_{k=p}^{\infty}(k+p\alpha\zeta)(1+k\lambda+p\lambda)^n b_k |a_{k,1}||a_{k,2}| \leq p(1-\alpha\zeta) \tag{33}$$

$$\sum_{k=p}^{\infty}\frac{(k+p\alpha\zeta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\zeta)}|a_{k,1}||a_{k,2}| \leq 1. \tag{34}$$

Applying Cauchy-Schwarz inequality, we have

$$\sum_{k=p}^{\infty}\frac{(1+k\lambda+p\lambda)^n b_k}{p}\frac{\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\sqrt{|a_{k,1}||a_{k,2}|} \leq 1. \tag{35}$$

From (34)and (35), we have

$$\frac{(k+p\alpha\zeta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\zeta)}|a_{k,1}||a_{k,2}|$$
$$\leq \frac{(1+k\lambda+p\lambda)^n b_k}{p}\frac{\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\sqrt{|a_{k,1}||a_{k,2}|}$$

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\frac{(1-\alpha\zeta)}{(k+p\alpha\zeta)}. \tag{36}$$

We know that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{(1+k\lambda+p\lambda)^n b_k\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}}, \tag{37}$$

from equation (36) and (37), we have

$$\frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{(1+k\lambda+p\lambda)^n b_k\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}} \leq \frac{\sqrt{(k+p\alpha\beta)(k+p\alpha\gamma)}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}\frac{(1-\alpha\zeta)}{(k+p\alpha\zeta)} \tag{38}$$

$$\zeta \leq \frac{(k+p\alpha\beta)(k+p\alpha\gamma)(1+k\lambda+p\lambda)^n b_k - kp(1-\alpha\beta)(1-\alpha\gamma)}{\alpha[p^2(1-\alpha\beta)(1-\alpha\gamma)+(k+p\alpha\beta)(k+p\alpha\gamma)(1+k\lambda+p\lambda)^n]}. \tag{39}$$

Now defining a function $\Psi(k)$ by

$$\Psi(k) = \frac{(k+p\alpha\beta)(k+p\alpha\gamma)(1+k\lambda+p\lambda)^n b_k - kp(1-\alpha\beta)(1-\alpha\gamma)}{\alpha[p^2(1-\alpha\beta)(1-\alpha\gamma)+(k+p\alpha\beta)(k+p\alpha\gamma)(1+k\lambda+p\lambda)^n]}, (k \geq p),$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\zeta \leq \frac{(1+\alpha\beta)(1+\alpha\gamma)(1+2p\lambda)^n b_p - (1-\alpha\beta)(1-\alpha\gamma)}{\alpha[(1-\alpha\beta)(1-\alpha\gamma)+(1+\alpha\beta)(1+\alpha\gamma)(1+2p\lambda)^n b_p]},$$

which completes the proof.

Theorem 8. Let the functions $f_j(z)(j = 1, 2)$ defined by $f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}|z^k \quad (j = 1, 2)$ be in the class $\mathcal{M}_g^n(\lambda, \alpha, \beta)$ then the function $h(z)$ defined by

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} (|a_{k,1}|^2 |a_{k,2}|^2) z^k \tag{40}$$

belongs to the class $\mathcal{M}_g^n(\lambda, \alpha, \gamma)$ where

$$\gamma \leq \frac{1}{\alpha} \left(\frac{p(1 + \alpha\beta)^2 (1 + 2p\lambda)^n - 2(1 - \alpha\beta)^2)}{2(1 - \alpha\beta)^2 + p(1 + \alpha\beta)^2 (1 + 2p\lambda)^n} \right). \tag{41}$$

Proof. Noting that

$$\sum_{k=p}^{\infty} \left[\frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\beta)} \right]^2 |a_{k,j}|^2$$

$$\leq \left[\sum_{k=p}^{\infty} \frac{(k+p\alpha\beta)(1+k\lambda+p\lambda)^n b_k}{p(1-\alpha\beta)} |a_{k,j}| \right]^2 \leq 1, (j = 1, 2). \tag{42}$$

For $f_j(z) \in \mathcal{M}_g^n(\lambda, \alpha, \beta)(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \frac{1}{2} \left[\frac{(k + p\alpha\beta)(1 + k\lambda + p\lambda)^n}{(1 - \alpha\beta)} \right]^2 (|a_{k,1}|^2 + |a_{k,2}|^2 \leq 1. \tag{43}$$

Therefore we have to find the largest γ such that

$$\sum_{k=p}^{\infty} \left[\frac{(k + p\alpha\gamma)(1 + k\lambda + p\lambda)^n b_k}{p(1 - \alpha\gamma)} \right] (|a_{k,1}|^2 + |a_{k,2}|^2) \leq 1, (k \geq p). \tag{44}$$

From equation (44) and (43) we have

$$\left[\frac{(k + p\alpha\gamma)(1 + k\lambda + p\lambda)^n}{(1 - \alpha\gamma)} \right] \leq \frac{1}{2} \left[\frac{(k + p\alpha\beta)(1 + k\lambda + p\lambda)^n}{(1 - \alpha\beta)} \right]^2, (k \geq p. \tag{45}$$

$$\gamma \leq \frac{1}{\alpha} \left(\frac{(k + p\alpha\beta)^2 (1 + k\lambda + p\lambda)^n - 2k(1 - \alpha\beta)^2)}{2p(1 - \alpha\beta)^2 + (k + p\alpha\beta)^2 (1 + k\lambda + p\lambda)^n} \right), (k \geq p). \tag{46}$$

Now defining a function $\Psi(k)$ by

$$\Psi(k) \frac{1}{\alpha} \left(\frac{(k + p\alpha\beta)^2 (1 + k\lambda + p\lambda)^n - 2k\ (1 - \alpha\beta)^2)}{2p(1 - \alpha\beta)^2 + (k + p\alpha\beta)^2 (1 + k\lambda + p\lambda)^n} \right), (k \geq p)$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\gamma \leq \frac{1}{\alpha}\left(\frac{p(1+\alpha\beta)^2(1+2p\lambda)^n - 2(1-\alpha\beta)^2)}{2(1-\alpha\beta)^2 + p(1+\alpha\beta)^2(1+2p\lambda)^n}\right).$$

6. Subordination Properties

If f and g are analytic functions in $\mathcal{U}$, we say that f is subordinate to g, written symbolically as follows, $f \prec g$ in $\mathcal{U}$ or $f(z) \prec g(z), \quad z \in \mathcal{U}$.

If there exists a function w which is analytic in $\mathcal{U}$ with $w(0) = 0, |w(z)| < 1 \quad (z \in \mathcal{U})$ such that

$$f(z) = g(w(z)) \quad (z \in \mathcal{U}). \tag{47}$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathcal{U} \Rightarrow f(0) = g(0), f(\mathcal{U}) \subset g(\mathcal{U}) \tag{48}$$

In particular, if one function g is univalent in $\mathcal{U}$ we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathcal{U} \Rightarrow f(0) = g(0), f(\mathcal{U}) \subset g(\mathcal{U}). \tag{49}$$

Let

$$\phi : C^2 \to C$$

be a function and let h be univalent in $\mathcal{U}$. If J is analytic function in $\mathcal{U}$ and satisfied the differential subordination

$$\phi(J(z), J^{'}(z)) \prec h(z)$$

then J is called a solution of the differential subordination

$$\phi(J(z), J^{'}(z)) \prec h(z).$$

The univalent function q is called a dominant of the solution of the differential subordination $J \prec q$.

Lemma 2. Let $q(z) \neq 0$ be univalent in $\mathcal{U}$. Let θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

1. $Q(z)$ is starlike univalent in $\mathcal{U}$ and

2. $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathcal{U}$.

If J is analytic function in $\mathcal{U}$ and

$$\theta(J(z)) + zJ'(z)\phi(J(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{50}$$

then

$$J(z) \prec q(z)$$

and q is the best dominant.

Lemma 3. Let $w, \gamma \in C$ and ϕ is convex and univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $Re\{w\phi(z) + \gamma\} > 0$ for all $z \in \mathcal{U}$. If q is analytic in $\mathcal{U}$ with $q(0) = 1$ and

$$q(z) + \frac{zq^{'}(z)}{wq(z) + \gamma} \prec \phi(z), \quad (z \in \mathcal{U}) \tag{51}$$

then $q(z) \prec \phi(z)$ is the best dominant.

Theorem 9. Let $q(z) \neq 0$ be univalent in $\mathcal{U}$ such that $\frac{zq^{'}(z)}{q(z)}$ is starlike univalent in $\mathcal{U}$ and

$$\Re\left\{1 + \frac{\epsilon}{\gamma}q(z) + \frac{zq^{''}(z)}{q^{'}(z)} - \frac{zq^{'}(z)}{q(z)}\right\} > 0, \quad (\epsilon, \gamma \in C, \gamma \neq 0). \tag{52}$$

If $f \in \Sigma_p$ satisfies the subordination

$$\epsilon\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} + \gamma\left[1 + \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{''}}{[\mathcal{D}_\lambda^n(f*g)(z)]^{'}} - \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]}\right] \prec \epsilon q(z) + \gamma\frac{zq^{'}(z)}{q(z)} \tag{53}$$

then $\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} \prec q(z)$ and q is the best dominant.

Proof. To apply lemma

$$f(z) = \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} = \frac{-pz^{-p} + \sum_{k=0}^{\infty} k(1+p\lambda+k\lambda)^n b_k a_k z^k}{z^{-p} + \sum_{k=0}^{\infty} k(1+p\lambda+k\lambda)^n b_k a_k z^k}$$

$$\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} = \frac{-p + \sum_{k=0}^{\infty} k(1+p\lambda+k\lambda)^n b_k a_k z^{k+p}}{1 + \sum_{k=0}^{\infty} k(1+p\lambda+k\lambda)^n b_k a_k z^{k+p}} \quad (n \in N_0; p \in N) \tag{54}$$

$\phi(w) = w$ and $\phi(w) = \frac{\gamma}{w}, \gamma \neq 0$. It can be easily observed that J is analytic in $\mathcal{U}$, θ ia analytic in C, ϕ is an analytic in $C/\{0\}$ and $\phi(w) \neq 0$.

By simple computation we get

$$\frac{zJ^{'}(z)}{J(z)} = 1 + \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{''}}{[\mathcal{D}_\lambda^n(f*g)(z)]^{'}} - \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} \tag{55}$$

which yields

$$\epsilon J(z) + \gamma \frac{zJ^{'}(z)}{J(z)} \prec \epsilon q(z) + \gamma \frac{zq^{'}(z)}{q(z)}. \tag{56}$$

That is

$$\theta(J(z)) + zJ'(z)\phi(J(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)). \tag{57}$$

Now by letting

$$\begin{aligned} Q(z) &= zq^{'}(z)\phi(q(z)) = \frac{\gamma z q^{'}(z)}{q(z)} \\ h(z) &= \theta(q(z)) + Q(z) = \epsilon q(z) + \frac{\gamma z q^{'}(z)}{q(z)} \end{aligned} \tag{58}$$

we find Q_i starlike univalent in $\mathcal{U}$ and that

$$\Re\left\{\frac{zh^{'}(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{\epsilon}{\gamma}q(z) + \frac{zq^{''}(z)}{q^{'}(z)} - \frac{zq^{'}(z)}{q(z)}\right\} > 0. \tag{59}$$

Hence by lemma, $\frac{z[\mathcal{D}_\lambda^n((f*g))(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} \prec q(z)$ and q is the best dominant.
Corollary 1. If $f \in \Sigma_p$ and assume that holds, then

$$1 + \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{''}}{[\mathcal{D}_\lambda^n(f*g)(z)]^{'}} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)} \tag{60}$$

implies that $\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} \prec \frac{1+Az}{1+Bz}$, $\quad -1 \le B < A \le 1$ and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By setting $\epsilon = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 9 , then we can obtain the result.
Corollary 2. If $f \in \Sigma_p$ and assume that holds, then

$$1 + \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{''}}{[\mathcal{D}_\lambda^n(f*g)(z)]^{'}} \prec e^{\alpha z} + \alpha z \tag{61}$$

implies that $\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]} \prec e^{\alpha z}$, $|\alpha| < \pi$ and $e^{\alpha z}$ is the best dominant.

Proof. The proof follows, by setting $\epsilon = \gamma = 1$ and $q(z) = e^{\alpha z}$, $(|\alpha| < \pi)$ in Theorem 9.

Theorem 10. Let $w, \gamma \in C$ and ϕ be convex and univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $\Re\{w\phi(z) + \gamma\} > 0$ for all $z \in \mathcal{U}$. If $f(z) \in \Sigma_p$ satisfies the subordination

$$\frac{1 + \gamma + \left(\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^n}{[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}\right) - (\frac{w}{p}+1)\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}{[\mathcal{D}_\lambda^n(f*g)(z)]}}{w - \gamma\left(\frac{p[\mathcal{D}_\lambda(f*g)(z)]}{z[\mathcal{D}_\lambda^n(f*g)(z)]^{'}}\right)} \prec \phi(z) \tag{62}$$

then

$$-\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]'}{p[\mathcal{D}_\lambda^n(f*g)(z)]} \prec \phi(z)$$

and ϕ is the best dominant.

Proof. Our aim to apply lemma setting

$$q(z) = -\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]'}{p[\mathcal{D}_\lambda^n(f*g)(z)]} = \frac{p+\sum_{k=p+1}^{\infty} k(k\lambda+p\lambda+1)^n b_k a_k z^{k+p}}{p+\sum_{k=p+1}^{\infty} p(k\lambda+p\lambda+1)^n b_k a_k z^{k+p}} \quad (n \in N_0; p \in N). \tag{63}$$

It can be easily observed that q is analytic in $\mathcal{U}$ and $q(0) = 1$. By Simple computation we get

$$\frac{zq'(z)}{q(z)} = 1 + \frac{z[D_\lambda^n(f*g)(z)]^n}{[\mathcal{D}_\lambda^n(f*g)(z)]'} - \frac{z[\mathcal{D}_\lambda^n(f*g)(z)]'}{[\mathcal{D}_\lambda^n(f*g)(z)]} \tag{64}$$

which yields,

$$q(z) + \frac{zq'(z)}{wq(z)+\gamma} \prec \phi(z), \quad (z \in \mathcal{U}). \tag{65}$$

Hence by lemma,

$$-\frac{z[\mathcal{D}_\lambda^n(f*g)(z)]'}{p[\mathcal{D}_\lambda^n(f*g)(z)]} \prec \phi(z)$$

and ϕ is the best dominant.

7.Concluding Remarks

In fact, by specializing the parameters λ, and by appropriately selecting the $g(z)$ (or fixing the coefficients b_n) as presented in the Examples 1 to 6 one would eventually lead us further to new results for the class of functions (defined analogously to the class $\mathcal{M}_g^n(\lambda,\alpha,\beta)$.)

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