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CERTAIN CLASS OF HARMONIC STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS DEFINED BY LINEAR OPERATOR

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Abstract. In this paper, we introduced a new class of complex-valued harmonic functions with respect to symmetric points by using linear operator. We obtain coefficient conditions, extreme points, distortion bounds, convex combinations for this family of harmonic univalent functions.

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1. Introduction

Denote by H the family of functions

$$f = h + \overline{g} \ , \tag{1}$$

which are analytic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$. So that f is normalized by $f(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in H$, we may express the analytic functions h and g in the forms

$$h(z) = z + \sum_{k=2}^{\infty} a_k \, z^k, \ \ g(z) = \sum_{k=1}^{\infty} b_k \, z^k \ , \ |b_1| < 1 \ . \tag{2}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in H is that $|h^{'}(z)| > |g^{'}(z)|$ in H (see [4]).
Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \ |b_1| < 1. \tag{3}$$

We denote $\overline{H}$ the subclass of H consists harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k},\ |b_1| < 1. \tag{4}$$

Let the Hadamard product (or convolution) of two power series $\Phi(z) = z + \sum_{k=2}^{\infty} \phi_k\, z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \psi_k\, z^k$ be defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k\, \psi_k\ z^k = (\Psi * \Phi)(z).$$

For positive real values of $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$; $j = 1, 2, ..., s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [18, p.19])

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} . \frac{z^k}{k!}$$

$$(q \le s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0;\ \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta + 1) .... (\theta + \nu - 1) & (\nu \in \mathbb{N};\ \theta \in \mathbb{C}). \end{cases} \tag{5}$$

Corresponding to the function $h_{q,s}(\alpha_1, \beta_1; z) = h(\alpha_1, ...\alpha_q; \beta_1 ..., \beta_s; z)$, defined by

$$\begin{aligned} h_{q,s}(\alpha_1, \beta_1; z) &= z {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \\ &= z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k, \end{aligned} \tag{6}$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} .... (\alpha_q)_{k-1}}{(\beta_1)_{k-1} ... (\beta_s)_{k-1} (k-1)!}. \tag{7}$$

In [11] El-Ashwah and Aouf defined the operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z)$ as follows:

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1) f(z) = z + \sum_{k=2}^{\infty} \left[\tfrac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) a_k z^k + \sum_{k=1}^{\infty} \overline{\left[\tfrac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) b_k\, z^k}, \tag{8}$$

where $m \in \mathbb{N}_0$, $\ell \geq 0$ and $\lambda \geq 0$.

We note that when $\ell = 0$, the operator $I^{m,0}_{q,s,\lambda}(\alpha_1, \beta_1)f(z) = D^m_\lambda(\alpha_1, \beta_1)f(z)$ was studied by Selvaraj and Karthikeyan [17].

We also note that:

(i) $I^{0,\ell}_{q,s,\lambda}f(z) = H_{q,s}(\alpha_1, \beta_1)f(z)$ (see Dziok and Srivastava [9,10]);

(ii) For $q = s+1$, $\alpha_i = 1(i = 1, ..., s+1)$ and $\beta_j = 1(j = 1, ..., s)$, we get the operator $I^m_{\lambda,\ell}$ (see Catas [5]);

(iii) For $q = s+1$, $\alpha_i = 1(i = 1, ..., s+1)$, $\beta_j = 1(j = 1, ..., s)$, $\lambda = 1$ and $\ell = 0$, we obtain the Salagean operator D^m (see Salagean [16]);

(iv) For $q = s+1$, $\alpha_i = 1(i = 1, ..., s+1)$, $\beta_j = 1(j = 1, ..., s)$ and $\lambda = 1$, we get the operator I^m_ℓ(see Cho and Srivastava [6] and Cho and Kim [7]).

(v) For $q = s+1, \alpha_i = 1(i = 1, ..., s+1), \beta_j = 1(j = 1, ..., s)$ and $\ell = 0$, we obtain the operator D^m_λ(see Al-Oboudi [3]).

By specializing the parameters $m, \lambda, \ell, q, s, \alpha_i(i = 1, .., q)$ and $\beta_j(j = 1, ..., s)$, we obtain:

(i) $I^{m,\ell}_{2,1,\lambda}(n+1, 1; 1)f(z) = I^{m,\ell}_\lambda(n)f(z) = z + \sum\limits_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \dfrac{(n+1)_{k-1}}{(1)_{k-1}} a_k z^k$

$$+ \sum_{k=1}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}} \overline{b_k\, z^k}, \ (n > -1);$$

(ii) $I^{m,\ell}_{2,1,\lambda}(a, 1; c)f(z) = I^{m,\ell}_\lambda(a; c)f(z) = z + \sum\limits_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \dfrac{(a)_{k-1}}{(c)_{k-1}} a_k z^k$

$$+ \sum_{k=1}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(a)_{k-1}}{(c)_{k-1}} \overline{b_k\, z^k}, \ (a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}^-_0);$$

(iii) $I^{m,\ell}_{2,1,\lambda}(2, 1; n+1)f(z) = I^{m,\ell}_{\lambda,n}f(z) = z + \sum\limits_{k=2}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \dfrac{(2)_{k-1}}{(n+1)_{k-1}} a_k z^k$

$$+ \sum_{k=1}^{\infty}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(2)_{k-1}}{(n+1)_{k-1}} \overline{b_k\, z^k}, \quad (n \in \mathbb{Z}; p \in \mathbb{N}; n > -1).$$

Motivated by Jahangiri et al. [12,13] and Ahuja and Jahangiri [1], we define a new subclass $HS^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$ of H that are starlike with respect symmetric points.

Definition 1. For $0 \leq \gamma < 1$ and $z = re^{i\theta} \in U$, we let $HS^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$ a subclass of H of the form $f = h + \overline{g}$ given by (3) and satisfying the analytic criteria

$$Re\left\{\frac{2z\left(I^{m,\ell}_{q,s,\lambda}(\alpha_1, \beta_1)f(z)\right)'}{z'\left[I^{m,\ell}_{q,s,\lambda}(\alpha_1, \beta_1)f(z) - I^{m,\ell}_{q,s,\lambda}(\alpha_1, \beta_1)f(-z)\right]}\right\} > \gamma, \tag{9}$$

where $I^{m,\ell}_{q,s,\lambda}(\alpha_1, \beta_1)f(z)$ is defined by (1.8) and $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$.

We also let $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma) = HS_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma) \cap \overline{H}$.

The family $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ is of special interest because for suitable choices of $q,\ s,\ [\alpha_1],\ [\beta_1]$ and $m,\ \ell,\ \lambda$. We note that
(i) $\overline{HS}_{s^*}^{\lambda,\ell,0}(q,s,[\alpha_1,\beta_1],\gamma) = \overline{HS}_s([\alpha_1],\gamma)$, which was studied by Murugusundaramoorthy et al. [14];
(ii) $\overline{HS}_{s^*}^{1,0,m}(2,1,[1,1],\gamma) = \overline{SH}_s(m,\alpha)$, which were studied by AL-Khal and Al-Kharsani [2].

Also, We note that
(i) $\overline{HS}_{s^*}^{\lambda,\ell,m}(2,1,[1;1],\gamma) = \overline{HS}_{s^*}^{m}(\lambda,\ell,\gamma)$

$$= \left\{ f(z) \in \overline{H} : Re\left(\frac{2z\ \left(I_{\lambda,\ell}^m f(z)\right)'}{z'\ \left[I_{\lambda,\ell}^m f(z) - I_{\lambda,\ell}^m f(-z)\right]} \right) > \gamma \right\}; \tag{10}$$

(ii) $\overline{HS}_{s^*}^{1,\ell,m}(2,1,[1;1],\gamma) = \overline{HS}_{s^*}^{m}(\ell,\gamma)$

$$= \left\{ f(z) \in \overline{H} : Re\left(\frac{2z\ \left(I_{\ell}^m f(z)\right)'}{z'\ \left[I_{\ell}^m f(z) - I_{\ell}^m f(-z)\right]} \right) > \gamma \right\}; \tag{11}$$

(iii) $\overline{HS}_{s^*}^{\lambda,0,m}(2,1,[1;1],\gamma) = \overline{HS}_{s^*}^{m}(\lambda,\gamma)$

$$= \left\{ f(z) \in \overline{H} : Re\left(\frac{2z\ \left(D_{\lambda}^m f(z)\right)'}{z'\ \left[D_{\lambda}^m f(z) - D_{\lambda}^m f(-z)\right]} \right) > \gamma \right\}; \tag{12}$$

(iv) $\overline{HS}_{s^*}^{\lambda,\ell,m}(2,1,[n+1,1;1],\gamma) = \overline{HS}_{s^*}^{m}(\lambda,\ell,n,\gamma)$

$$= \left\{ f(z) \in \overline{H} : Re\left(\frac{2z\ \left(I_{\lambda}^{m,\ell}(n) f(z)\right)'}{z'\ \left[I_{\lambda}^{m,\ell}(n) f(z) - I_{\lambda}^{m,\ell}(n) f(-z)\right]} \right) > \gamma \right\}; \tag{13}$$

(v) $\overline{HS}_{s^*}^{\lambda,\ell,m}(2,1,[a,1;c],\gamma) = \overline{HS}_{s^*}^{m,\lambda}(\ell,a,c,\gamma)$

$$= \left\{ f(z) \in \overline{H} : Re\left(\frac{2z\ \left(I_{\lambda}^{m,\ell}(a;c)) f(z)\right)'}{z'\ \left[I_{\lambda}^{m,\ell}(a;c) f(z) - I_{\lambda}^{m,\ell}(a;c) f(-z)\right]} \right) > \gamma \right\}. \tag{14}$$

Remarks 1. (i) If the co-analytic part of $f = h + \overline{g}$ is zero, $q = s+1,\ m = \ell = 0,\ \lambda = 1,\ \alpha_i = 1\,(i = 1,...,q)$ and $\beta_j = 1\,(j = 1,...,s)$ then $HS_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$

turns out to be the class $S_s^*(\gamma)$ of starlike functions with respect to symmetric points which was introduced by Sakaguchi [15];
(ii) If the co-analytic part of $f = h + \overline{g}$ is zero, $q = s + 1,\ m = \lambda = 1,\ \ell = 0,\ \alpha_i = 1\,(i = 1, ..., q)$ and $\beta_j = 1\,(j = 1, ..., s)$ then $HS_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$ turns out to be the class $K_s(\alpha)$ of convex functions with respect to symmetric points which was introduced by Das and Singh [8].

In this paper, we have obtained the coefficient conditions for the classes $HS_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$ and $\overline{HS}_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$. Further a representation theorem, inclusion properties and distortion bounds for the class $\overline{HS}_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$ are also established.

2. Coefficient characterization

Unless otherwise mentioned, we assume throughout this paper that $q, s \in \mathbb{N}$, $\alpha_1, ..., \alpha_q,\ \beta_1, ..., \beta_s \in \mathbb{R}^+,\ a_1 = 1$ and $0 \leq \gamma < 1$. We begin with a sufficient condition for functions in $HS_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$.

Theorem 1. *Let $f = h + \overline{g}$ be given by* (1.3)*. Furthermore, let*

$$\sum_{k=2}^{\infty} \frac{\left[2k - \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1 + \ell + \lambda(k-1)}{1 + \ell}\right]^m \Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1 + \ell + \lambda(k-1)}{1 + \ell}\right]^m \Gamma_k(\alpha_1)|b_k| \leq 1 \quad (15)$$

where $\Gamma_k(\alpha_1)$ be defined by (1.7)*. Then f is sense-preserving, harmonic univalent in U and $f \in HS_{s^*}^{\lambda,\ell,m}\left(q, s, [\alpha_1, \beta_1], \gamma\right)$.*

Proof. According the condition (9), we only need to show that if (15) holds, then

$$Re\left\{\frac{2z\ \left(I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)\right)'}{z'\ \left[I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z) - I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(-z)\right]}\right\} = Re\frac{A(z)}{B(z)} > \gamma,$$

where

$$A(z) = 2z\ \left(I_{q,s,\lambda}^{m,\ell}(\alpha_1, \beta_1)f(z)\right)' = 2z'\left[z + \sum_{k=2}^{\infty} k\left[\frac{1 + \ell + \lambda(k-1)}{1 + \ell}\right]^m \Gamma_k(\alpha_1)a_k z^k + \sum_{k=1}^{\infty} k\left[\frac{1 + \ell + \lambda(k-1)}{1 + \ell}\right]^m \Gamma_k(\alpha_1)\overline{b_k\, z^k}\right]$$

and

$$B(z) = z' \left[I^{m,\ell}_{q,s,\lambda}(\alpha_1,\beta_1)f(z) - I^{m,\ell}_{q,s,\lambda}(\alpha_1,\beta_1)f(-z) \right]$$
$$= z' \left[2z + \sum_{k=2}^{\infty} \left[1-(-1)^k\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) a_k z^k \right.$$
$$\left. + \sum_{k=1}^{\infty} \left[1-(-1)^k\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) \overline{b_k\, z^k} \right].$$

Using the fact that $Re\{w(z)\} > \gamma$ if and only if $|1-\gamma+w| > |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| > 0. \tag{16}$$

Substituting for $A(z)$ and $B(z)$ in (2.2) and by using (2.1), we obtain

$$\left| 2(2-\gamma)z + \sum_{k=2}^{\infty} \left[2k+(1-\gamma)(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) a_k\, z^k \right.$$
$$\left. + \sum_{k=1}^{\infty} \left[2k-(1-\gamma)(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) \overline{b_k\, z^k} \right|$$
$$- \left| -2\gamma\, z + \sum_{k=2}^{\infty} \left[2k-(1+\gamma)(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) a_k\, z^k \right.$$
$$\left. + \sum_{k=1}^{\infty} \left[2k+(1+\gamma)(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) \overline{b_k\, z^k} \right|$$

$$\geq 4(1-\gamma)|z| - 2\sum_{k=2}^{\infty} \left[2k-\gamma(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\, |a_k|\; |z|^k$$
$$-2\sum_{k=1}^{\infty} \left[2k+\gamma(1-(-1)^k)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\, |b_k|\; |z|^k$$

$$= 4(1-\gamma)|z| \left[1 - \sum_{k=2}^{\infty} \frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\, |a_k|\; |z|^{k-1} \right.$$
$$\left. - \sum_{k=1}^{\infty} \frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\, |b_k|\; |z|^{k-1} \right].$$

$$\geq 4(1-\gamma)\left[1-\sum_{k=2}^{\infty}\frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\,|a_k|\right.$$
$$\left.-\sum_{k=1}^{\infty}\frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\,|b_k|\right]\geq 0.$$

This last expression is non-negative by (2.1).

The harmonic univalent functions

$$f(z) = z+\sum_{k=2}^{\infty}\frac{2(1-\gamma)}{\left[2k-\gamma(1-(-1)^k)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)}X_k\,z^k$$
$$+\sum_{k=1}^{\infty}\frac{2(1-\gamma)}{\left[2k+\gamma(1-(-1)^k)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)}\overline{Y_k}\,\overline{z}^k, \tag{17}$$

where $\sum\limits_{k=2}^{\infty}|X_k|+\sum\limits_{k=1}^{\infty}|Y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $HS_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ because

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2\left(1-\gamma\right)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_k|$$
$$+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2\left(1-\gamma\right)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\,|b_k|=\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1.$$

This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions f to be in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$.

Theorem 2. *Let $f=h+\overline{g}$ be given by* (1.4). *Then $f\in\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2\left(1-\gamma\right)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_k|$$
$$+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2\left(1-\gamma\right)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_k|\leq 1, \tag{18}$$

where $\Gamma_k(\alpha_1)$ *be defined by* (1.7).

Proof. Since $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma) \subset HS_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f(z)$ of the form (1.4), we notice that the condition

$$Re\left\{\frac{2z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)\right)'}{z'\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(z)-I_{q,s,\lambda}^{m,\ell}(\alpha_1,\beta_1)f(-z)\right]}\right\}>\gamma$$

is equivalent to

$$Re\left\{\left[2(1-\gamma)-\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)a_kz^{k-1}\right.\right.$$
$$\left.-\frac{\overline{z}}{z}\sum_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)b_k\,\overline{z}^{k-1}\right]\Big/$$
$$\left[2-\sum_{k=2}^{\infty}(1-(-1)^k)\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)a_kz^{k-1}\right.$$
$$\left.\left.+\frac{\overline{z}}{z}\sum_{k=1}^{\infty}(1-(-1)^k)\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)b_k\,\overline{z}^{k-1}\right]\right\}>0. \tag{19}$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\le z=r<1$, we must have

$$\left\{2(1-\gamma)-\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)a_kr^{k-1}\right.$$
$$-\sum_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)b_k\,r^{k-1}\Big/$$
$$2-\sum_{k=2}^{\infty}(1-(-1)^k)\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)a_kr^{k-1}$$
$$\left.+\sum_{k=1}^{\infty}(1-(-1)^k)\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)b_k\,r^{k-1}\right\}>0. \tag{20}$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f(z) \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ and so the proof of Theorem 2 is completed.

By specialyzing the parameters $\lambda,\ \ell,\ m,\ q,\ s,\ \alpha$'s and β's, we obtain the following corollaries.

Corollary 1. *For* $f = h + \overline{g} \in \overline{HS}_{s^*}^{m}(\lambda,\ell,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m |a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m |b_k| \leq 1.$$

Corollary 2. *For* $f = h + \overline{g} \in \overline{HS}_{s^*}^{m}(\ell,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{k+\ell}{1+\ell}\right]^m |a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{k+\ell}{1+\ell}\right]^m |b_k| \leq 1.]$$

Corollary 3. *For* $f = h + \overline{g} \in \overline{HS}_{s^*}^{m}(\lambda,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} [1+\lambda(k-1)]^m |a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} [1+\lambda(k-1)]^m |b_k| \leq 1. \tag{21}$$

Corollary 4. *For* $f = h + \overline{g} \in \overline{HS}_{s^*}^{m}(\lambda,\ell,n,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}} |a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(n+1)_{k-1}}{(1)_{k-1}} |b_k| \leq 1.$$

Corollary 5. *For* $f = h + \overline{g} \in \overline{HS}^{m,\lambda}_{s^*}(\ell, a, c, \gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{\left[2k - \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(a)_{k-1}}{(c)_{k-1}} |a_k|$$
$$+ \sum_{k=1}^{\infty} \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \frac{(a)_{k-1}}{(c)_{k-1}} |b_k| \le 1.$$

3. Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of the class $\overline{HS}^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$ denoted by *clco* $\overline{HS}^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$.

Theorem 3. *A function* $f_k \in clco\, \overline{HS}^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$ *if and only if* $f_k(z)$ *can be expressed in the form*

$$f_k(z) = \sum_{k=1}^{\infty} \left[X_k\, h_k(z) + Y_k g_k(z)\right], \tag{22}$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{2(1-\gamma)}{\left[2k - \gamma\left(1 - (-1)^k\right)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} z^k \; (k \ge 2),$$

and

$$g_k(z) = z + \frac{2(1-\gamma)}{\left[2k + \gamma\left(1 - (-1)^k\right)\right] \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} \overline{z}^k \; (k \ge 1),$$

$$X_k \ge 0,\; Y_k \ge 0,\; \sum_{k=1}^{\infty} (X_k + Y_k) = 1.$$

In particular, the extreme points of the class $\overline{HS}^{\lambda,\ell,m}_{s^*}(q, s, [\alpha_1, \beta_1], \gamma)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions $f_k(z)$ of the form (22), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} X_k z^k + \sum_{k=1}^{\infty} \frac{2(1-\gamma)}{\left[2k+\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} Y_k \overline{z}^k.$$

Then by using Theorem 2, we have

$$\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)|b_k|$$

$$= \sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) \cdot \frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} X_k$$

$$+ \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1) \cdot \left(\frac{2(1-\gamma)}{\left[2k+\gamma\left(11-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)} Y_k\right)$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \le 1,$$

and so $f_k \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$.

Conversely, if $f_k \in clco\ \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$. Setting

$$X_k = \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\,|a_k|\ (k \ge 2),$$

and

$$Y_k = \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)\,|b_k| \ (k \geq 1).$$

We obtain $f_k(z) = \sum\limits_{k=1}^{\infty} [X_k\, h_k(z) + Y_k g_k(z)]$ as required.

The following theorem gives the distortion theorem for functions in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$ which yields a covering result for this class.

Theorem 4. *Let the functions* $f(z)$ defined by (4) be in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \leq (1 + |b_1|)\, r + \frac{1}{\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \left\{\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right\} r^2,$$

and

$$|f(z)| \geq (1 - |b_1|)\, r - \frac{1}{\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \left\{\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right\} r^2.$$

The result is sharp.

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z) \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$. Taking the absolute value of f we have

$$|f(z)| \leq (1 + |b_1|)\, r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\, r^k \leq (1 + |b_1|) r + \, r^{j2} \sum_{k=2}^{\infty} (|a_k| + |b_k|)$$

$$\leq (1 + |b_1|) r + \frac{(1-\gamma)}{\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \sum_{k=2}^{\infty} \frac{\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)}{1-\gamma} (|a_k| + |b_k|) r^2$$

$$\leq (1 + |b_1|) r + \frac{(1-\gamma) r^2}{\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \frac{\left[2k - \gamma\left(1 - (-1)^k\right)\right]}{4(1-\gamma)} |a_k| \right.$$

$$\left. + \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{4(1-\gamma)} |b_k| \right\} \cdot \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)$$

$$= (1+|b_1|)r + \frac{(1-\gamma)r^2}{2\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} |a_k| \right.$$

$$\left. + \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)} |b_k| \right\} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m \Gamma_k(\alpha_1)$$

$$\leq (1+|b_1|)r + \frac{(1-\gamma)r^2}{2\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \left(1 - \frac{1+\gamma}{1-\gamma}|b_1|\right)$$

$$= (1+|b_1|)r + \frac{1}{\left[\frac{1+\ell+\lambda}{1+\ell}\right]^m \Gamma_2(\alpha_1)} \left[\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right] r^2.$$

The proof of Theorem 4 is completed.

4. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k, \ |b_1| < 1 \tag{23}$$

and

$$G(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \overline{z}^k \ (A_k \geq 0; \ B_k \geq 0) \tag{24}$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \overline{z}^k. \tag{25}$$

Using this definition, we show that the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$ is closed under convolution.

Theorem 5. *For $0 \leq \mu \leq \gamma < 1$, let $f \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$ and $G \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \mu)$. Then*

$f * G \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma) \subset \overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \mu)$.

Proof. Let the function $f(z)$ defined by (4.1) be in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \gamma)$ and let the function $G(z)$ defined by (4.2) be in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q, s, [\alpha_1, \beta_1], \mu)$.

Then the convolution $f * G$ is given by (4.3). We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 2. For $G \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\mu)$ we note that $0 \leq A_k \leq 1$ and $0 \leq B_k \leq 1$. Now, for the convolution function $f * G$ we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[2k-\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_k|A_k \\
&\qquad+\sum_{k=1}^{\infty}\left[2k+\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_k|B_k \\
\leq\ &\sum_{k=2}^{\infty}\left[2k-\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_k| \\
&\qquad+\sum_{k=1}^{\infty}\left[2k+\gamma\left(1-(-1)^k\right)\right]\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_k| \\
\leq\ &2(1-\gamma),
\end{aligned}$$

since $0 \leq \mu \leq \gamma < 1$ and $f \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$. Therefore $f * G \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma) \subset \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\mu)$, since the above inequality bounded by $2(1-\gamma)$ while $2(1-\gamma) \leq 2(1-\mu)$. This completes the proof of Theorem 5.

Now, we show that the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ is closed under convex combinations of its members.

Theorem 6. *The class* $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, ...,$ let $f_i \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty}|a_{k_i}|\,z^k + \sum_{k=1}^{\infty}|b_{k_i}|\,\overline{z}^k\,(a_{k_i} \geq 0;\ b_{k_i} \geq 0;\ z \in U).$$

Then by using Theorem 2, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_{k_i}| \\
&+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_{k_i}| \leq 1. \qquad (26)
\end{aligned}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i\, f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\, |a_{k_i}|\right) z^k + \sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} t_i\, |b_{k_i}|)\overline{z}^k. \tag{27}$$

Then, by using (4.4), we have

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\left(\sum_{i=1}^{\infty} t_i\,|a_{k_i}|\right)$$
$$+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\left(\sum_{i=1}^{\infty} t_i\,|b_{k_i}|\right)$$
$$=\sum_{i=1}^{\infty} t_i\left[\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_{k_i}|\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_{k_i}|\right]$$
$$\leq \sum_{i=1}^{\infty} t_i = 1, \text{ this is the necessary and sufficient condition given by (2.4)}$$

and so $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in \overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$. This completes the proof of Theorem 6.

5. Properties of certain integral operator

Finally, we study properties of certain integral operator.

Theorem 7. *Let the functions* $f(z)$ defined by (4) be in the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$ *and let* c be a real number such that $c > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \tag{28}$$

belongs to the class $\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$.

Proof. From the representation of $F(z)$, it follows that

$$\begin{aligned}
F(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{h(t)+\overline{g(t)}\right\}dt \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty}a_k\,t^k\right)dt+\int_0^z \overline{t^{c-1}\left(\sum_{k=1}^{\infty}b_k t^k\right)dt}\right) \\
&= \frac{c+1}{z^c}\left(\int_0^z t^c dt-\sum_{k=2}^{\infty}a_k\int_0^z t^{c+k-1}dt+\sum_{k=1}^{\infty}b_k\int_0^z \overline{t^{c+k-1}dt}\right) \\
&= z-\sum_{k=2}^{\infty}A_k z^k+\sum_{k=1}^{\infty}B_k\overline{z}^k,
\end{aligned}$$

where $A_k=\frac{c+1}{c+k}a_k,\ B_k=\frac{c+1}{c+k}b_k$. Therefore

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\frac{c+1}{c+k}|a_k| \\
&\quad+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)\frac{c+1}{c+k}|b_k| \\
&\quad<\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|a_k| \\
&\quad+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m\Gamma_k(\alpha_1)|b_k|\le 1.
\end{aligned}$$

Since $f(z)\in\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$, therefore from Theorem 2, we have $F(z)\in\overline{HS}_{s^*}^{\lambda,\ell,m}(q,s,[\alpha_1,\beta_1],\gamma)$.

Remarks 2. Putting $m=0$ in our results we obtain the results obtained by Murugusundaramoorthy et al. [14].

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