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GRAPHICAL LECTURES OF SOME KING TYPE OPERATORS

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Abstract. In this paper are graphicaly represented the images of some King type operators which have as fixed points the test functions $\{e_0, e_1\}$; $\{e_0, e_2\}$ or $\{e_1, e_2\}$; both in finite case, of Bernstein type, but also in infinite case, of Mirakjan type. The graphics overlap on the same axis, more operators for the same functions, and so, we can make comparisions about the quality of uniform approximation.

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1. Introduction

The ideea of introducing of uniform approximation operators, linears and positive of Bernstein type, having an approximation order better than the classical operators, belongs to J. P. King which had introduce an operator which reproduce the test functions e_0 and e_2. In this way of generalization of fundamental polynomials, the King operator approximate better the continous functions defined on the unit interval with values in $\left(0; \frac{1}{3}\right)$ [1]. On this direction, P. Braica, Ovidiu T. Pop and A. Indrea brought in [2] an operator of King type, which reproduce e_1 and e_2 of Bernstein type and Ovidiu T. Pop, D. Bărbosu and P. Braica approached the same problem starting with Szász-Mirakjan operators and introducing the Mirakjan operators of King type, reproducing e_0 and e_2, and also operators of PB type, reproducing e_1 and e_2 [3]. The comparision problem of this three type of operators presume also to involves tracking their behavior from graphical perspective for particular continous functions.

Definition 1.*Let $n \in \mathbb{N}$; the operators $B_n : C[0;1] \to C[0;1]$, $f \to (B_n f)$ defined by*

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

where $x \in [0;1]$ and with $C[0;1]$ we denote the Banach space of all continous functions defined on [0;1]; are called Bernstein operators.

Definition 2. *Let* $(r_n(x))_n \in \mathbb{N}$ *be a sequence of finite functions from* $C[0;1]$*, with* $r_n(x) \in [0;1]$*. The operators* $V_n : C[0;1] \to C[0;1]$ *defined by:*

$$(V_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right)$$

where $x \in [0;1]$ *and* $f \in C[0;1]$*,* $k \geq 0$*, are called King type operators.*
Observation 3. *If we choose*

$$r_n(x) = r_n^*(x) = \begin{cases} x^2, & \text{if } n = 1 \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{if } n = 2, 3, ... \end{cases}$$

then $(V_n e_o)(x) = e_0(x)$ and $(V_n e_2)(x) = e_2(x)$ for any $x \in [0;1]$.
Definition 4. *Let* $n \in \mathbb{N}$*; the operators* $B_n^* : C\left[\frac{1}{n_0-1}; 1\right] \to \left[\frac{1}{n_0-1}; 1\right]$ *where* $n \geq n_0$*,* $n_0 \in \mathbb{N}$ *fixed, defined by:*

$$(B_n^* f)(x) = \frac{(n-1)x}{nx-1}\left(1 - \frac{1}{n}\right)^{-n} \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{1}{n}\right)^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

are called operators of PB type.
Observation 5. *T*he PB operators, uniformely approximate any continous function on $\left[\frac{1}{n_0-1}; 1\right]$, with $n \geq n_0$, $n_o \in \mathbb{N}$ fixed and $(B_n^* e_1)(x) = e_1(x)$; $(B_n^* e_2)(x) = e_2(x)$ for any $x \in \left[\frac{1}{n_0-1}; 1\right]$ and $n \geq n_0$.
Definition 6. *The operators* $S_n : C_2([0, +\infty)) \to C([0, +\infty))$ *defined by:*

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{nx^k}{k!} f\left(\frac{k}{n}\right)$$

for any $x \in [0, +\infty)$ *and any* $n \in \mathbb{N}$*, where*

$$C_2([0, +\infty)) = \left\{ f \in C[0, +\infty) : \exists \lim_{x \to +\infty} \frac{f(x)}{(1+x)^2} \in \mathbb{R} \right\}$$

are called Szász-Mirakjan operators.
Observation 7. *The Szász-Mirakjan operators, uniformely approximate any continous function from* $C_2([0, +\infty))$ *on the compact intervals* $[0; b]$*, where* $b \in \mathbb{R}$ *is fixed.*
Definition 8. *Let* $n \in \mathbb{N}$ *and* $x \in [0; +\infty)$*. The operators defined by:* $A_n : C_2([0, +\infty)) \to C_2([0, +\infty))$

$$(A_n f)(x) = e^{\frac{1-\sqrt{1+4n^2x^2}}{2}} \sum_{k=0}^{\infty} \frac{(\sqrt{1+4n^2x^2} - 1)^k}{2^k k!} f\left(\frac{k}{n}\right)$$

for any $n \in \mathbb{N}$, $f \in C_2([0,+\infty))$ *and any* $x \in [0;+\infty)$, *are called Mirakjan operators of King type.*

Observation 9. *The Mirakjan operators of King type, uniformely approximate any continous function from* $C_2([0,+\infty))$ *on the compact intervals* $[0;b]$, *where* $b \in \mathbb{R}$ *is fixed and also this type of operators, reproduce the test functions* e_0 *and* e_2.
Definition 10. *Let* $n_0 \in \mathbb{N}$ *fixed and* $n \geq n_0$. *The operators defined by:* $S_n^* : C_2\left(\left[\frac{1}{n_0-1};+\infty\right)\right) \to C_2\left(\left[\frac{1}{n_0-1};+\infty\right)\right)$

$$\left(S_n^* f\right)(x) = \frac{nx}{nx-1} e^{1-nx} \sum_{k=0}^{\infty} \frac{(nx-1)^k}{k!} f\left(\frac{k}{n}\right)$$

are called Mirakjan operators of PB type.
Observation 11. *The Mirakjan operators of PB type, uniformely approximate any continous function from* $C_2\left(\left[\frac{1}{n_0-1};+\infty\right)\right)$ *on the compact intervals* $[\frac{1}{n_0-1};b]$, $b > \frac{1}{n_0-1}$ *where ,* $b \in \mathbb{R}$ *is fixed and also this type of operators, reproduce the test functions* e_1 *and* e_2.

Definition 12. *Let* $\alpha, \beta \in \mathbb{R}_+$ *fixed, with* $0 < \alpha \leq \beta$ *and* $n \in \mathbb{N}$. *The operators defined by:* $P_n^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$

$$\left(P_n^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right),$$

$f \in C([0,1])$ *and any* $x \in [0;1]$, *are called Bernstein-Stancu type operators.*

Definition 13. *Let* $\alpha, \beta \in \mathbb{R}_+$ *fixed, with* $0 < \alpha \leq \beta$ *and* $n \in \mathbb{N}$. *The operators defined by:* $B_n^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$

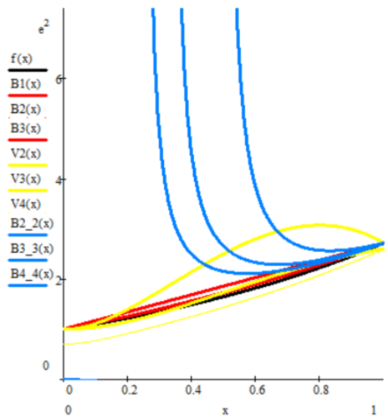
$$\left(B_n^{(\alpha,\beta)} f\right)(x) = \frac{1}{n^n} \sum_{k=0}^{n} \binom{n}{k} ((n+\beta x)-\alpha)^k (n+\alpha-(n+\beta)x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right),$$

for any $x \in [0;1]^*$ *and* $f \in C([0,1])$ *are called the generalized Stancu operator of Bernstein type.*
Observation 14. *The generalized Stancu operator of Bernstein type, uniformely approximate any continous function* $f \in C[0;1]$; *they have positive coefficients just for* $x \in \left[\frac{\alpha}{n+\beta};\frac{n+\alpha}{n+\beta}\right]$ *and they reproduce like the classical Bernstein operators the test functions* e_0 *and* e_1.

2.The expresions of represented images

On figure 1, on the same graphic $x \in [0;1]$, $y \in [0,e^2]$ are plotted the following:

- *with black* $f:[0;1] \to \mathbb{R}$, $f(x) = exp(x) = e^x$; $\forall x \in [0;1]$
- *with red*, the classical Bernstein itterations:
  $(B_1 exp)(x) = (1-x) + xe; \forall x \in [0;1]$
  $(B_2 exp)(x) = (1-x)^2 + 2x(1-x)e^{\frac{1}{2}} + ex^2; \forall x \in [0;1]$
  $(B_3 exp)(x) = (1-x)^3 + 3x(1-x)^2 e^{\frac{1}{3}} + 3x^2(1-x)e^{\frac{2}{3}} + ex^3; \forall x \in [0;1]$
- *with yellow* :
  $(V_2 exp)(x) = \left(\frac{3}{2} - \sqrt{x^2+\frac{1}{4}}\right)^2 + 2\left(\frac{3}{2} - \sqrt{x^2+\frac{1}{4}}\right)\left(\sqrt{x^2+\frac{1}{4}} - \frac{1}{2}\right)e^{\frac{1}{2}} +$
  $\left(\sqrt{x^2+\frac{1}{4}} - \frac{1}{2}\right)^2 e; \forall x \in [0;1]$
  $(V_3 exp)(x) = \left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)^3 + 3\left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)^2\left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)e^{\frac{1}{3}} +$
  $3\left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)\left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)^2 e^{\frac{2}{3}} + \left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)^3 e; \forall x \in [0;1]$
  $(V_4 exp)(x) = \left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^4 + 4\left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^3\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)e^{\frac{1}{4}}$
  $+ 6\left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^2\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)^2 e^{\frac{2}{4}} + 4\left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)$
  $\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)^3 e^{\frac{3}{4}}\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)^4 e; \forall x \in [0;1]$
- *w*ith blue:
  $(B_2^* exp)(x) = \frac{4x}{2x-1}\left((1-x)^2 + (1-x)(2x-1)e^{\frac{1}{2}} + \left(x-\frac{1}{2}\right)^2 e\right);$
  $(B_3^* exp)(x) = \frac{27x}{4(3x-1)}\left((1-x)^3 + 3(1-x)^2\left(x-\frac{1}{3}\right)e^{\frac{1}{3}}\right.$
  $\left. +3(1-x)\left(x-\frac{1}{3}\right)^2 e^{\frac{2}{3}} + \left(x-\frac{1}{3}\right)^3 e\right)$
  $(B_4^* exp)(x) = \frac{256x}{27(4x-1)}((1-x)^4 + 4(1-x)^3\left(x-\frac{1}{4}\right)e^{\frac{1}{4}} + 6(1-x)^2\left(x-\frac{1}{4}\right)^2 e^{\frac{2}{4}} +$
  $4(1-x)\left(x-\frac{1}{4}\right)^3 e^{\frac{3}{4}} + \left(x-\frac{1}{4}\right)^4 e);$



On figure 2, on the same graphic $x \in [0;1]$, $y \in [0, e^2]$ are plotted the images of the operators from figure 1 but this time for: $f : [0;1] \to \mathbb{R}$, $f(x) = ln(1+x)$

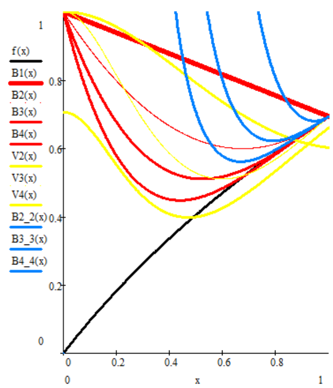
- *with black* $f : [0;1] \to \mathbb{R}$, $f(x) = ln(1+x)$; $\forall x \in [0;1]$
- *with red*, the classical Bernstein itterations:
  $(B_1 \ln(1+\cdot))(x) = (1-x) + x\ln(1+1)$; $\forall x \in [0;1]$
  $(B_2 \ln(1+\cdot))(x) = (1-x)^2 + 2x(1-x)\ln\frac{3}{2} + x^2\ln 2$; $\forall x \in [0;1]$
  $(B_3 \ln(1+\cdot))(x) = (1-x)^3 + 3x(1-x)^2\ln\frac{4}{3} + 3x^2(1-x)\ln\frac{5}{3} + x^3\ln 2$;
  $(B_4 \ln(1+\cdot))(x) = (1-x)^4 + 4x(1-x)^3\ln\frac{5}{4} + 6x^2(1-x)^2\ln\frac{6}{4} + 4x^3(1-x)\ln\frac{7}{4} + x^4\ln 2$;

- *with yellow, the King type operator*: $(V_2 \ln(1+\cdot))(x) = \left(\frac{3}{2} - \sqrt{x^2+\frac{1}{4}}\right)^2 + 2\left(\frac{3}{2} - \sqrt{x^2+\frac{1}{4}}\right)\left(\sqrt{x^2+\frac{1}{4}} - \frac{1}{2}\right)\ln\frac{3}{2} + \left(\sqrt{x^2+\frac{1}{4}} - \frac{1}{2}\right)^2\ln 2$;
  $(V_3 \ln(1+\cdot))(x) = \left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)^3 + 3\left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)^2\left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)\ln\frac{4}{3} + 3\left(\frac{5}{4} - \sqrt{\frac{3}{2}x^2+\frac{1}{16}}\right)\left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)^2\ln\frac{5}{3} + \left(\sqrt{\frac{3}{2}x^2+\frac{1}{16}} - \frac{1}{4}\right)^3\ln 2$;

  $(V_4 \ln(1+\cdot))(x) = \left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^4 + 4\left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^3\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)\ln\frac{5}{4} + 6\left(\frac{7}{6} - \sqrt{\frac{4}{3}x^2+\frac{1}{16}}\right)^2\left(\sqrt{\frac{4}{3}x^2+\frac{1}{36}} - \frac{1}{6}\right)^2\ln\frac{6}{4} +$

$$4\left(\tfrac{7}{6}-\sqrt{\tfrac{4}{3}x^2+\tfrac{1}{16}}\right)\left(\sqrt{\tfrac{4}{3}x^2+\tfrac{1}{36}}-\tfrac{1}{6}\right)^3\ln\tfrac{7}{4}+\left(\sqrt{\tfrac{4}{3}x^2+\tfrac{1}{36}}-\tfrac{1}{6}\right)^4\ln 2;$$

- *with* blue, the PB type operator:
  $(B_2^*\ln(1+\cdot))(x)=\frac{4x}{2x-1}\left((1-x)^2+(1-x)(2x-1)\ln\frac{3}{2}+\left(x-\frac{1}{2}\right)^2e\right);$
  $(B_3^*\ln(1+\cdot))(x)=\frac{27x}{4(3x-1)}((1-x)^3+3(1-x)^2\left(x-\frac{1}{3}\right)\ln\frac{4}{3}+3(1-x)\left(x-\frac{1}{3}\right)^2\ln\frac{5}{3}+\left(x-\frac{1}{3}\right)^3\ln 2)$
  $(B_4^*\ln(1+\cdot))(x)=\frac{256x}{27(4x-1)}((1-x)^4+4(1-x)^3\left(x-\frac{1}{4}\right)\ln\frac{5}{4}+6(1-x)^2\left(x-\frac{1}{4}\right)^2\ln\frac{6}{4}+4(1-x)\left(x-\frac{1}{4}\right)^3\ln\frac{7}{4}+\left(x-\frac{1}{4}\right)^4\ln 2);$



On figure 3, we compare the images of Mirakjan type operators for $f:[0;10]\to\mathbb{R}$, $f(x)=e^{-x}$, $f\in C_2\left([0;+\infty)\right)$, because

$$\lim_{x\to\infty}\frac{f(x)}{1+x^2}=0\in\mathbb{R}$$

is finite.

- *with black*

$$(S_2exp)(x)=e^{-2x}\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{(2x)^k}{k!}e^{-\frac{k}{2}}\right)$$

$$(S_3exp)(x)=e^{-3x}\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{(3x)^k}{k!}e^{-\frac{k}{3}}\right)$$

$$(S_4exp)(x) = e^{-4x} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(4x)^k}{k!} e^{-\frac{k}{4}} \right)$$

$\forall x \in [0; 10]$.

- *with yellow the Mirakjan operator of King type*

$$(A_2exp)(x) = e^{\frac{1-\sqrt{16x^2+1}}{2}} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(\sqrt{16x^2+1}-1)^k}{2^k k!} e^{-\frac{k}{2}} \right)$$

$$(A_3exp)(x) = e^{\frac{1-\sqrt{36x^2+1}}{2}} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(\sqrt{36x^2+1}-1)^k}{2^k k!} e^{-\frac{k}{3}} \right)$$

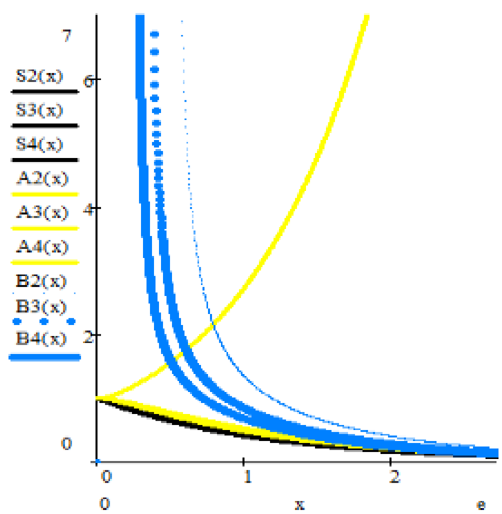
$$(A_4exp)(x) = e^{\frac{1-\sqrt{64x^2+1}}{2}} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(\sqrt{64x^2+1}-1)^k}{2^k k!} e^{-\frac{k}{4}} \right)$$

- *with blue, the Mirakjan operator of* e_1e_2 *type*:

$$(S_2^*exp)(x) = \frac{2x}{2x-1} e^{1-2x} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(2x-1)^k}{k!} e^{-\frac{k}{2}} \right)$$

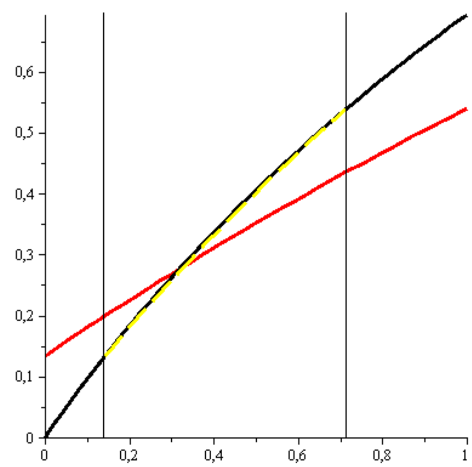
$$(S_3^*exp)(x) = \frac{3x}{3x-1} e^{1-3x} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(3x-1)^k}{k!} e^{-\frac{k}{3}} \right)$$

$$(S_4^*exp)(x) = \frac{4x}{4x-1} e^{1-4x} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{(4x-1)^k}{k!} e^{-\frac{k}{4}} \right)$$



On figure 4, we compare graphical images of classical Stancu operator with generalized Stancu operator of Bernstein type, $\alpha = 1$, $\beta = 2$, for the logarithmic function $f : [0, 1] \to \mathbb{R}$, $f(x) = ln(1 + x)$, $x \in [0, 1]$. We choose $\alpha = 1$, $\beta = 3$ and we take $n_0 = 2$. For $n = 4$, we plot

- with black: $f : [0, 1] \to \mathbb{R}$, $f(x) = \ln(1 + x)$
- with red: $\left(P_4^{(1,3)} \ln(1 + \cdot)\right)(x) = (1 - x)^4 \ln\left(\frac{8}{7}\right) + 4x(1 - x)^3 \ln\left(\frac{9}{7}\right) + 6x^2(1 - x) \ln\left(\frac{10}{7}\right) + 4x^3(1 - x) \ln\left(\frac{11}{7}\right) + x^4 \ln\left(\frac{12}{7}\right)$, for $x \in [0, 1]$
- with yellow: $\left(Q_4^{(1,3)} \ln(1 + \cdot)\right)(x) = \frac{1}{256}(5 - 7x)^4 \ln\left(\frac{8}{7}\right) + 4(5 - 7x)^3(7x - 1)\frac{1}{256} \ln\left(\frac{9}{7}\right) + 6(5 - 7x)^2(7x - 1)^2\frac{1}{256} \ln\left(\frac{10}{7}\right) + 4(5 - 7x)(7x - 1)^3\frac{1}{256} \ln\left(\frac{11}{7}\right) + (7x - 1)^4\frac{1}{256} \ln\left(\frac{12}{7}\right)$, for $x \in \left[\frac{1}{7}, \frac{5}{7}\right]$



From the graphical lectures of the considered operators, we can observe the following:

- The operator B_m^* has a better approximation for exponential function on the boundary of 1, than the classical Bernstein operator.
- The Mirakjan operator of type e_1, e_2 has a better approximation for exponential function e^{-x}, than the classical Mirakjan operator.
- The operator $Q_n^{\alpha,\beta}$ has a quick approximation for logarithmic function, than the classical Stancu operator.

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