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STABILITY BY KRASNOSELSKII FIXED POINT THEOREM FOR NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract. We use Krasnoselskii's fixed point theorem to obtain boundedness and stability results about the zero solution of a neutral nonlinear differential equation with variable delays. A stability theorem with a necessary and sufficient condition is given. The results obtained here extend and improve the works of C. H. Jin and J. W. Luo [12] and also those of [5, 9, 15]. In the end we provide an example to illustrate our claim.

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1. Introduction

Certainly, the Liapunov direct method has been, for more than 100 years, the main tool for the study of stability properties of ordinary, functional and partial differential equations. Nevertheless, the application of this method to problems of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms $[3-5]$. Recently, T. A. Burton and T. Furumochi, B. Zhang and others investigators noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see $[1-13, 15]$). The fixed point theory does not only solve the problem on stability but has a significant advantage over Liapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [3]).

In this paper we consider the neutral nonlinear differential equation with variable delays

$$\begin{aligned} x'(t) &= -a(t)\,x(t-r_1(t)) + \frac{d}{dt}Q(t, x(t-r_1(t))) \\ &\quad + c(t)\,F(x(t-r_1(t)), x(t-r_2(t))) + b(t)\,G(x^{\gamma}(t-r_2(t))), \end{aligned} \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \ \ for \ t \in [m(0), 0],$$

where $\psi \in C([m(0), 0], \mathbb{R})$, $m_j(0) = \inf\{t - r_j(t), \ t \geq 0\}$, $m(0) = \min\{m_j(0), j = 1, 2\}$, $\gamma \in (0, 1)$ and γ is a quotient with odd positive integer denominator. Throughout this paper we assume that $a, b, c \in C(\mathbb{R}^+, \mathbb{R})$, and $r_1 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, $r_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2$. The functions $G(x)$, $Q(t, x)$ and $F(x, y)$ are locally Lipschitz continuous in x, x and in x and y, respectively. That is, there are positive constants L_1, L_2, L_3, L_4 so that if $|x|, |y|, |z|, |w| \leq 1$ then

$$\begin{gathered} |G(x) - G(y)| \leq L_1 |x - y|, \ \ |Q(t, x) - Q(t, y)| \leq L_2 |x - y|, \\ |F(x, y) - F(z, w)| \leq L_3 |x - z| + L_4 |y - w|. \end{gathered} \tag{1.2}$$

We also assume that

$$G(0) = Q(t, 0) = F(0, 0) = 0. \tag{1.3}$$

Our purpose here is to give, by using Krasnoselskii fixed point theorem, boundedness and stability results for the nonlinear neutral differential equation with variable delays (1.1).

Special cases of equation (1.1) have been previously considered and studied under various conditions. Particularly, T. A. Burton in [5] and B. Zhang in [15] have investigated the boundedness and stability of the linear equation

$$x'(t) = -a(t) x(t - r_1(t)).$$

In [9], T. A. Burton and T. Furumochi have studied the boundedness and the asymptotic stability by using Krasnoselskii fixed point theorem for the following equation

$$x'(t) = -a(t) x(t - r_1) + b(t) x^{\frac{1}{3}}(t - r_2(t)),$$

with $r_1 \geq 0$ is a constant and $a \in C(\mathbb{R}^+, (0, \infty))$. In the case $\gamma = 1/3$, $G(x) = x$, $Q(t, x) = 0$, and $c(t) = 0$, C. H. Jin and J. W. Luo in [12] studied, by means of Krasnoselskii's fixed point theorem, the boundedness and the stability of the zero solution, under appropriate conditions, of the following equation

$$x'(t) = -a(t) x(t - r_1(t)) + b(t) x^{\frac{1}{3}}(t - r_2(t)),$$

and generalized the results claimed previously by [5, 9, 15].

In Section 2, we present the inversion of neutral nonlinear differential equation (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [3, 14]. We present our main results on stability in Section 3 and at the end we provide an example to illustrate our claim.

2. Inversion of equation(1.1)

We have to invert equation (1.1). For this, we use the variation of parameter formula and rewrite the equation as an integral equation suitable for Krasnoselskii theorem.

Lemma 2.1. *Let $g : [m(0), \infty) \to \mathbb{R}^+$ be an arbitrary continuous function. Then x is a solution of* (1.1) *if and only if*

$$\begin{aligned}
x(t) &= \left(x(0) - Q(0, x(-r_1(0))) - \int_{-r_1(0)}^{0} g(u)\, x(u)\, du \right) e^{-\int_0^t g(u)du} \\
&+ Q(t, x(t - r_1(t))) + \int_{t-r_1(t)}^{t} g(u)\, x(u)\, du \\
&- \int_0^t e^{-\int_s^t g(u)du} g(s) \left(\int_{s-r_1(s)}^{s} g(u)\, x(u)\, du \right) ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} \left[g(s - r_1(s)) \left(1 - r_1'(s)\right) - a(s) \right] x(s - r_1(s))\, ds \\
&- \int_0^t e^{-\int_s^t g(u)du} Q(s, x(s - r_1(s)))\, g(s)\, ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} c(s)\, F(x(s - r_1(s)), x(s - r_2(s)))\, ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} b(s)\, G(x^{\gamma}(s - r_2(s)))\, ds. \qquad (2.1)
\end{aligned}$$

Proof. Let x be a solution of (1.1). Rewrite equation (1.1) as

$$\begin{aligned}
&\frac{d}{dt} \{x(t) - Q(t, x(t - r_1(t)))\} \\
&= -g(t) \{x(t) - Q(t, x(t - r_1(t)))\} \\
&+ \frac{d}{dt} \int_{t-r_1(t)}^{t} g(s)\, x(s)\, ds \\
&+ \left[g(t - r_1(t)) \left(1 - r_1'(t)\right) - a(t) \right] x(t - r_1(t)) \\
&- g(t)\, Q(t, x(t - r_1(t))) \\
&+ c(t)\, F(x(t - r_1(t)), x(t - r_2(t))) \\
&+ b(t)\, G(x^{\gamma}(t - r_2(t))).
\end{aligned}$$

Multiply both sides of the above equation by $e^{\int_0^t g(s)ds}$ and then integrate from 0 to

t to obtain

$$\begin{aligned}
x(t) &= \left(x(0) - Q(0, x(-r_1(0)))\right) e^{-\int_0^t g(s)ds} + Q(t, x(t - r_1(t))) \\
&+ \int_0^t e^{-\int_s^t g(u)du} d\left(\int_{s-r_1(s)}^s g(u)\, x(u)\, du\right) \\
&+ \int_0^t e^{-\int_s^t g(u)du} \left[g(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right] x(s - r_1(s))\, ds \\
&- \int_0^t e^{-\int_s^t g(u)du} Q(s, x(s - r_1(s)))\, g(s)\, ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} c(s)\, F(x(s - r_1(s)), x(s - r_2(s)))\, ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} b(s)\, G(x^\gamma(s - r_2(s)))\, ds. \qquad (2.2)
\end{aligned}$$

By performing an integration by parts, we have

$$\begin{aligned}
&\int_0^t e^{-\int_s^t g(u)du} d\left(\int_{s-r_1(s)}^s g(u)\, x(u)\, du\right) \\
&= -e^{-\int_0^t g(u)du} \int_{-r_1(0)}^0 g(u)\, x(u)\, du + \int_{t-r_1(t)}^t g(u)\, x(u)\, du \\
&- \int_0^t e^{-\int_s^t g(u)du} g(s) \left(\int_{s-r_1(s)}^s g(u)\, x(u)\, du\right) ds. \qquad (2.3)
\end{aligned}$$

Finally, substituting (2.3) into (2.2) ends the proof.

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the stability of the zero solution. For its proof we refer the reader to $[3, 14]$.

Theorem 2.1. (Krasnoselskii) *Let M be a closed convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into S such that*

(i) $x, y \in M$, implies $Ax + By \in M$,

(ii) A is continuous and AM is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there exists $z \in M$ with $z = Az + Bz$.

3. Stability by Krasnoselkii fixed point theorem

From existence theory, which can be found in [3], we conclude that for each continuous initial function $\psi : [m(0), 0] \to \mathbb{R}$, there is a continuous solution $x(t, 0, \psi)$

on an interval $[0,T)$ for some $T>0$ and $x(t,0,\psi)=\psi(t)$ on $[m(0),0]$. For stability definitions we refer to [3].

Theorem 3.1. *Suppose conditions* (1.2) *and* (1.3) *hold, and that there are constants* $\alpha\in(0,1)$, $k_1,k_2>0$ *and a function* $g\in C([m(0),\infty),\mathbb{R}^+)$ *such that for* $|t_2-t_1|\leq 1$,

$$\left|\int_{t_1}^{t_2}|b(u)|\,du\right|\leq k_1|t_1-t_2|, \tag{3.1}$$

and

$$\left|\int_{t_1}^{t_2}g(u)\,du\right|\leq k_2|t_1-t_2|, \tag{3.2}$$

while for $t\geq 0$

$$\begin{aligned} &L_2+\int_{t-r_1(t)}^{t}g(u)\,du+\int_0^t e^{-\int_s^t g(u)du}g(s)\left(\int_{s-r_1(s)}^{s}g(u)\,du\right)ds \\ &+\int_0^t e^{-\int_s^t g(u)du}\{\left|g(s-r_1(s))\left(1-r_1'(s)\right)-a(s)\right| \\ &+L_2g(s)+(L_3+L_4)|c(s)|+L_1|b(s)|\}ds\ \leq\ \alpha. \end{aligned} \tag{3.3}$$

If ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t,0,\psi)$ *of* (1.1) *on* $\mathbb{R}^+$ *with* $|x(t,0,\psi)|\leq 1$.

Proof. For $\alpha\in(0,1)$, find an appropriate $\delta>0$ such that

$$\left\{\left|1-\int_{-r_1(0)}^{0}g(u)\,du\right|+L_2\right\}e^{-\int_0^t g(s)ds}\delta+\alpha\leq 1.$$

Let $\psi:[m(0),0]\to\mathbb{R}$ be a given small bounded initial function with $\|\psi\|<\delta$. In the same context as in papers [3, 9, 12], let $h:[m(0),\infty)\to[1,\infty)$ be any strictly increasing and continuous function with $h(m(0))=1$, $h(s)\to\infty$ as $s\to\infty$, such that

$$\begin{aligned} &L_2+\int_{t-r_1(t)}^{t}g(u)h(u)/h(t)\,du \\ &+\int_0^t e^{-\int_s^t g(u)du}g(s)\left(\int_{s-r_1(s)}^{s}g(u)h(u)/h(t)\,du\right)ds \\ &+\int_0^t e^{-\int_s^t g(u)du}\left[\{\left|g(s-r_1(s))\left(1-r_1'(s)\right)-a(s)\right|\right. \\ &\left.+L_2g(s)+L_3|c(s)|\}h(s-r_1(s))+L_4|c(s)|h(s-r_2(s))\right]/h(t)\,ds\ \leq\ \alpha. \end{aligned} \tag{3.4}$$

Let $(S, |\cdot|_h)$ be the Banach space of continuous $\varphi : [m(0), \infty) \to \mathbb{R}$ with

$$|\varphi|_h := \sup_{t \geq m(0)} |\varphi(t)/h(t)| < \infty,$$

and define the set S_ψ by

$$S_\psi = \{\varphi \in S : |\varphi(t)| \leq 1 \text{ for } t \in [m(0), \infty) \text{ and } \varphi(t) = \psi(t) \text{ if } t \in [m(0), 0]\}.$$

Define the mappings $A, B : S_\psi \to S_\psi$ by

$$(A\varphi)(t) = \int_0^t e^{-\int_s^t g(u)du} b(s) G(\varphi^\gamma(s - r_2(s))) ds, \tag{3.5}$$

and

$$\begin{aligned}(B\varphi)(t) = {} & \left(x(0) - Q(0, \varphi(-r_1(0))) - \int_{-r_1(0)}^0 g(u)\varphi(u) du\right) e^{-\int_0^t g(s)ds} \\ & + Q(t, \varphi(t - r_1(t))) + \int_{t-r_1(t)}^t g(u)\varphi(u) du \\ & - \int_0^t e^{-\int_s^t g(u)du} g(s) \left(\int_{s-r_1(s)}^s g(u)\varphi(u) du\right) ds \\ & + \int_0^t e^{-\int_s^t g(u)du} \left[g(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right] \varphi(s - r_1(s)) ds \\ & - \int_0^t e^{-\int_s^t g(u)du} Q(s, \varphi(s - r_1(s))) g(s) ds \\ & + \int_0^t e^{-\int_s^t g(u)du} c(s) F(\varphi(s - r_1(s)), \varphi(s - r_2(s))) ds. \end{aligned} \tag{3.6}$$

That A maps S_ψ into itself can be deduced from condition (3.3).

We now show that $\varphi, \phi \in S_\psi$ implies that $A\varphi + B\phi \in S_\psi$. When doing this we see that also B maps S_ψ into itself by letting $\varphi = 0$ in the preceding sum. Now, let $\|\cdot\|$ be the supremum norm on $[m(0), \infty)$ of $\varphi \in S_\psi$ if φ is bounded. Note that if $\varphi, \phi \in S_\psi$ then

$$\begin{aligned} & |(A\varphi)(t) + (B\phi)(t)| \\ & \leq \left\{\left|1 - \int_{-r_1(0)}^0 g(u) du\right| + L_2\right\} e^{-\int_0^t g(s)ds} \|\psi\| + L_2\|\phi\| + \|\phi\| \int_{t-r_1(t)}^t g(u) du \\ & + \|\phi\| \int_0^t e^{-\int_s^t g(u)du} g(s) \left(\int_{s-r_1(s)}^s g(u) du\right) ds \end{aligned}$$

$$
\begin{aligned}
&+ \|\phi\| \int_0^t e^{-\int_s^t g(u)du} \left\{ \left| g\left(s - r_1(s)\right)\left(1 - r_1'(s)\right) - a(s) \right| \right\} ds \\
&+ \|\phi\| \int_0^t e^{-\int_s^t g(u)du} L_2 g(s)\, ds + \|\phi\| \int_0^t e^{-\int_s^t g(u)du} \left(L_3 + L_4\right) |c(s)|\, ds \\
&+ \|\varphi\|^{\gamma} \int_0^t e^{-\int_s^t g(u)du} L_1 |b(s)|\, ds \\
&\leq \left\{ \left| 1 - \int_{-r_1(0)}^{0} g(u)\, du \right| + L_2 \right\} e^{-\int_0^t g(s)ds} \delta + \alpha \leq 1.
\end{aligned}
$$

Next, we show that AS_ψ is equicontinuous. If $\varphi \in S_\psi$ and $0 \leq t_1 < t_2$ with $t_2 - t_1 < 1$, then

$$
\begin{aligned}
&\left| (A\varphi)(t_2) - (A\varphi)(t_1) \right| \\
&= \left| \int_0^{t_2} e^{-\int_s^{t_2} g(u)du} b(s)\, G\left(\varphi^{\gamma}\left(s - r_2(s)\right)\right) ds \right. \\
&\left. - \int_0^{t_1} e^{-\int_s^{t_1} g(u)du} b(s)\, G\left(\varphi^{\gamma}\left(s - r_2(s)\right)\right) ds \right| \\
&\leq \left| \int_{t_1}^{t_2} e^{-\int_s^{t_2} g(u)du} b(s)\, G\left(\varphi^{\gamma}\left(s - r_2(s)\right)\right) ds \right| \\
&+ \left| \int_0^{t_1} \left[e^{-\int_s^{t_2} g(u)du} - e^{-\int_s^{t_1} g(u)du} \right] b(s)\, G\left(\varphi^{\gamma}\left(s - r_2(s)\right)\right) ds \right| \\
&\leq L_1 \int_{t_1}^{t_2} e^{-\int_s^{t_2} g(u)du} d\left(\int_{t_1}^{s} |b(s)|\, ds \right) \\
&+ L_1 \left| e^{-\int_s^{t_2} g(u)du} - e^{-\int_s^{t_1} g(u)du} \right| e^{\int_0^{t_1} g(u)du} \int_0^{t_1} e^{-\int_s^{t_1} g(u)du} |b(s)|\, ds \\
&\leq L_1 \int_{t_1}^{t_2} |b(u)|\, du \left(1 + \int_{t_1}^{t_2} e^{-\int_s^{t_2} g(u)du} g(s)\, ds \right) + \alpha \left| e^{-\int_{t_1}^{t_2} g(u)du} - 1 \right| \\
&\leq 2L_1 \int_{t_1}^{t_2} |b(u)|\, du + \alpha \left| \int_{t_1}^{t_2} g(u)\, du \right| \leq \left(2L_1 k_1 + \alpha k_2\right) |t_2 - t_1|,
\end{aligned}
$$

by (1.2), (1.3) and (3.1) − (3.3). In the space $\left(S, |\cdot|_h\right)$, the set AS_ψ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem AS_ψ resides in a compact set.

Now we show that B is a contraction with respect to the norm $|\cdot|_h$ with constant α. Indeed,

$$
\begin{aligned}
&\left| (B\phi_1)(t) - (B\phi_2)(t) \right| / h(t) \\
&\leq \left| Q\left(t, \phi_1\left(t - r_1(t)\right)\right) - Q\left(t, \phi_2\left(t - r_1(t)\right)\right) \right| / h(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t-r_1(t)}^{t} g(u)\left|\phi_1(u)-\phi_2(u)\right|/h(t)\,du \\
&+\int_0^t e^{-\int_s^t g(u)du} g(s)\left(\int_{s-r_1(s)}^{s} g(u)\left|\phi_1(u)-\phi_2(u)\right|/h(t)\,du\right)ds \\
&+\int_0^t e^{-\int_s^t g(u)du}\left|g(s-r_1(s))\left(1-r_1'(s)\right)-a(s)\right| \\
&\times\left|\phi_1(s-r_1(s))-\phi_2(s-r_1(s))\right|/h(t)\,ds \\
&+\int_0^t e^{-\int_s^t g(u)du} g(s)\left|Q(s,\phi_1(s-r_1(s)))-Q(s,\phi_2(s-r_1(s)))\right|/h(t)\,ds \\
&+\int_0^t e^{-\int_s^t g(u)du}\left|c(s)\right|\left|F(\phi_1(s-r_1(s)),\phi_1(s-r_2(s)))\right. \\
&\left.-F(\phi_2(s-r_1(s)),\phi_2(s-r_2(s)))\right|/h(t)\,ds \\
&\leq\left|\phi_1-\phi_2\right|_h\left\{L_2+\int_{t-r_1(t)}^{t} g(u)h(u)/h(t)\,du\right. \\
&+\int_0^t e^{-\int_s^t g(u)du} g(s)\left(\int_{s-r_1(s)}^{s} g(u)h(u)/h(t)\,du\right)ds \\
&+\int_0^t e^{-\int_s^t g(u)du}\left[\left\{\left|g(s-r_1(s))\left(1-r_1'(s)\right)-a(s)\right|\right.\right. \\
&\left.\left.+L_2 g(s)+L_3\left|c(s)\right|\right\}h(s-r_1(s))+L_4\left|c(s)\right|h(s-r_2(s))\right]/h(t)\,ds\Big\} \\
&\leq\alpha\left|\phi_1-\phi_2\right|_h,
\end{aligned}
$$

by (1.2) and (3.4).

Finally, we need to show that A is continuous. Let $\epsilon>0$ be given and let $\varphi\in S_\psi$. Now x^γ is uniformly continuous on $[-1,1]$ so for a fixed $T>0$ with $4/h(T)<\epsilon$ there is a $\eta>0$ such that $|x_1-x_2|<\eta h(T)$ implies $|x_1^\gamma-x_2^\gamma|<\epsilon/2$. Thus for $|\varphi(t)-\phi(t)|<\eta h(t)$ and for $t>T$ we have

$$
\begin{aligned}
&\left|(A\varphi)(t)-(A\phi)(t)\right|/h(t) \\
&\leq(1/h(t))\int_0^t e^{-\int_s^t g(u)du}\left|b(s)\right|\left|G(\varphi^\gamma(s-r_2(s)))-G(\phi^\gamma(s-r_2(s)))\right|ds \\
&\leq L_1(1/h(t))\int_0^t e^{-\int_s^t g(u)du}\left|b(s)\right|\left|\varphi^\gamma(s-r_2(s))-\phi^\gamma(s-r_2(s))\right|ds \\
&\leq L_1(1/h(t))\left\{\int_0^T e^{-\int_s^t g(u)du}\left|b(s)\right|\left|\varphi^\gamma(s-r_2(s))-\phi^\gamma(s-r_2(s))\right|ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+2\int_T^t |b(s)|\, e^{-\int_s^t g(u)du} ds \Big\} \\
\leq\ & L_1 \{(\alpha\epsilon)/(2h(t)) + 2\alpha/h(T)\} \leq L_1 \{(\alpha\epsilon/2) + (2\alpha/h(T))\} < L_1\alpha\epsilon.
\end{aligned}
$$

The conditions of Krasnoselskii's theorem are satisfied with $M = S_\psi$ and there is a fixed point which solve our problem. This completes the proof.

Letting $r_1(t) = r_1$, a constant, and $g(t) = a(t + r_1)$ with $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, we have

Corollary 3.1. *Let* (1.2), (1.3), (3.1) *and* (3.2) *hold and* (3.3) *be replaced by*

$$
\begin{aligned}
&L_2 + \int_{t-r_1}^t a(u + r_1)\, du + \int_0^t e^{-\int_s^t a(u+r_1)du} a(s + r_1) \left(\int_{s-r_1}^s a(u + r_1)\, du\right) ds \\
&+ \int_0^t e^{-\int_s^t a(u+r_1)du} (L_2 a(s + r_1) + (L_3 + L_4)|c(s)| + L_1 |b(s)|)\, ds \leq \alpha. \qquad (3.7)
\end{aligned}
$$

If ψ is a given continuous initial function which is sufficiently small, then there is a solution $x(t, 0, \psi)$ of (1.1) *on $\mathbb{R}^+$ with $|x(t, 0, \psi)| \leq 1$.*

Letting $\gamma = 1/3$, $G(x) = x$, $Q(t, x) = 0$ and $c(t) = 0$, we have

Corollary 3.2. *Let* (3.1) *and* (3.2) *hold and* (3.3) *be replaced by*

$$
\begin{aligned}
&\int_{t-r_1(t)}^t g(u)\, du + \int_0^t e^{-\int_s^t g(u)du} g(s) \left(\int_{s-r_1(s)}^s g(u)\, du\right) ds \\
&+ \int_0^t e^{-\int_s^t g(u)du} \left\{\left|g(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right| + |b(s)|\right\} ds \leq \alpha. \qquad (3.8)
\end{aligned}
$$

If ψ is a given continuous initial function which is sufficiently small, then there is a solution $x(t, 0, \psi)$ of (1.1) *on $\mathbb{R}^+$ with $|x(t, 0, \psi)| \leq 1$.*

Remark 3.1. *The corollary* 3.2 *improves Theorem* 2.1 *in* [12].

Theorem 3.2. *Let* (1.2), (1.3) *and* (3.1) − (3.3) *hold and assume that*

$$
\int_0^t e^{-\int_s^t g(u)du} |b(s)|\, ds \to 0 \text{ as } t \to \infty, \qquad (3.9)
$$

If ψ is a given continuous initial function which is sufficiently small, then (1.1) *has a solution $x(t, 0, \psi) \to 0$ as $t \to \infty$ if and only if*

$$
\int_0^t g(s)\, ds \to \infty \text{ as } t \to \infty. \qquad (3.10)
$$

Proof. First, suppose that (3.10) holds. We set

$$
N = \sup_{t \geq 0} \left\{e^{-\int_0^t g(s)ds}\right\}. \qquad (3.11)
$$

All of the calculations in the proof of Theorem 3.1 hold with $h(t) = 1$ when $|\cdot|_h$ is replaced by the supremum norm $\|\cdot\|$.
For $\varphi \in S_\psi$, (1.2) and (1.3) implies

$$|(A\varphi)(t)| \leq L_1 \int_0^t e^{-\int_s^t g(u)du} |b(s)| \, ds =: q(t), \tag{3.12}$$

where $q(t) \to 0$ as $t \to \infty$ by (3.9).

Add to S_ψ the condition that $\varphi \in S_\psi$ implies that $\varphi(t) \to 0$ as $t \to \infty$. We can see that for $\varphi \in S_\psi$ then $(A\varphi)(t) \to 0$ as $t \to \infty$ by (3.12), and $(B\phi)(t) \to 0$ as $t \to \infty$ by (1.3) and (3.10). Since AS_ψ has been shown to be equicontinuous, A maps S_ψ into a compact subset of S_ψ (see [3], Theorem 1.2.2 on p. 20). By Krasnoselskii's theorem there is an $x \in S_\psi$ with $Ax + Bx = x$. As $x \in S_\psi$, $x(t) \to 0$ as $t \to \infty$.

Conversely, suppose (3.10) fails. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim\limits_{n\to\infty} \int_0^{t_n} g(u)\, du = l$ for some $l \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} g(s)\, ds \leq J,$$

for all $n \geq 1$. To simplify the expression, we define

$$\omega(s) = \left|g(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right| + (L_3 + L_4)|c(s)| + L_1 |b(s)| + g(s)\left(L_2 + \int_{s-r_1(s)}^{s} g(u)\, du\right),$$

for all $s \geq 0$. By (3.3), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} g(u)du} \omega(s)\, ds \leq \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s g(u)du} \omega(s)\, ds \leq \alpha e^{\int_0^{t_n} g(u)du} \leq J.$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s g(u)du} \omega(s)\, ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s g(u)du} \omega(s)\, ds = \lambda,$$

for some $\lambda \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s g(u)du}\omega(s)\,ds < \delta_0/4N,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 N e^J + \alpha \leq 1$.
We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\psi(t_m) - Q(t_m, \psi(t_m - r_1(t_m))) - \int_{t_m - r_1(t_m)}^{t_m} g(s)\psi(s)\,ds \geq \frac{1}{2}\delta_0.$$

It follows from (3.5) and (3.6) with $x(t) = (Ax)(t) + (Bx)(t)$ that for $n \geq m$

$$\begin{aligned}
&\left| x(t_n) - Q(t_n, x(t_n - r_1(t_n))) - \int_{t_n - r_1(t_n)}^{t_n} g(s)x(s)\,ds \right| \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} g(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} g(u)du}\omega(s)\,ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} g(u)du} - e^{-\int_0^{t_n} g(u)du}\int_{t_m}^{t_n} e^{\int_0^s g(u)du}\omega(s)\,ds \\
&= e^{-\int_{t_m}^{t_n} g(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} g(u)du}\int_{t_m}^{t_n} e^{\int_0^s g(u)du}\omega(s)\,ds\right) \\
&\geq e^{-\int_{t_m}^{t_n} g(u)du}\left(\frac{1}{2}\delta_0 - N\int_{t_m}^{t_n} e^{\int_0^s g(u)du}\omega(s)\,ds\right) \\
&\geq \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} g(u)du} \geq \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad (3.13)
\end{aligned}$$

On the other hand, if the solution of (1.1) $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$, since $t_n - r_1(t_n) \to \infty$ as $n \to \infty$ and (3.3) holds, we have

$$x(t_n) - Q(t_n, x(t_n - r_1(t_n))) - \int_{t_n - r_1(t_n)}^{t_n} g(s)x(s)\,ds \to 0 \text{ as } n \to \infty,$$

which contradicts (3.13). Hence condition (3.10) is necessary in order that (1.1) has a solution $x(t, 0, \psi) \to 0$ as $t \to \infty$. The proof is complete.

Letting $r_1(t) = r_1$, a constant, and $g(t) = a(t + r_1)$ with $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, we have

Corollary 3.3. *Let* (1.2), (1.3), (3.1), (3.2) *and* (3.9) *hold and* (3.3) *be replaced by* (3.7). *If ψ is a given continuous initial function which is sufficiently small, then* (1.1) *has a solution $x(t, 0, \psi) \to 0$ as $t \to \infty$ if and only if*

$$\int_0^t g(s)\,ds \to \infty \text{ as } t \to \infty.$$

For the case $\gamma = 1/3,\ G(x) = x,\ Q(t,x) = 0$ and $c(t) = 0$, we have

Corollary 3.4. *Let* (3.1), (3.2) *and* (3.9) *hold and* (3.3) *be replaced by* (3.8). *If* ψ *is a given continuous initial function which is sufficiently small, then* (1.1) *has a solution* $x(t,0,\psi) \to 0$ *as* $t \to \infty$ *if and only if*

$$\int_0^t g(s)\,ds \to \infty \text{ as } t \to \infty.$$

Remark 3.2. *The corollary* 3.4 *improves Theorem* 2.2 *in* [12].

Example 3.1. Let

$$\begin{aligned} x'(t) &= -a(t)\,x(t - r_1(t)) + \frac{d}{dt}Q(t, x(t - r_1(t))) \\ &\quad + c(t)\,F(x(t - r_1(t)), x(t - r_2(t))) + b(t)\,G(x^{\gamma}(t - r_2(t))), \end{aligned} \tag{3.14}$$

where $\gamma = 1/3,\ G(x) = \sin(x),\ Q(t,x) = (1/16)\sin(x),\ F(x,y) = \sin(x+y)$, $r_1(t) = 0.172t$, $r_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(t) = 0.928/(0.828t + 1)$, $b(t) = 1/\left(8(t+1)^2\right)$, $c(t) = \sin(t)/(16t + 16)$.
Then for any small continuous initial function ψ, every solution $x(t,0,\psi)$ of the nonlinear neutral differential equation (3.14) goes to 0 as $t \to \infty$.

Clearly $G(0) = Q(t,0) = F(0,0) = 0$ and $G(x)$, $Q(t,x)$ and $F(x,y)$ are locally Lipschitz continuous in x, x and in x and y, respectively. Let $|x|, |y|, |z|, |w| \le 1$, then

$$|G(x) - G(y)| = |\sin(x) - \sin(y)| \le |x - y|,$$

and

$$|Q(t,x) - Q(t,y)| = (1/16)\,|\sin(x) - \sin(y)| \le (1/16)\,|x - y|,$$

and

$$|F(x,y) - F(x,y)| = |\sin(x+y) - \sin(z+w)| \le |x - z| + |y - w|.$$

Choosing $g(t) = 1/(t+1)$, we have

$$\begin{aligned} &\textstyle\int_{t-r_1(t)}^t g(s)\,ds = \int_{0.828t}^t 1/(s+1)\,ds = \ln\left(\frac{t+1}{0.828t+1}\right) < 0.189, \\ &\textstyle\int_0^t e^{-\int_s^t g(u)du} g(s)\left(\int_{s-r_1(s)}^s g(u)\,du\right) ds < 0.189, \\ &\int_0^t e^{-\int_s^t g(u)du}\left|g(s - r_1(s))\left(1 - r_1'(s)\right) - a(s)\right| ds \\ &< \frac{0.1}{0.828}\int_0^t e^{-\int_s^t \frac{1}{u+1}du}\frac{1}{s+1}ds < 0.121, \\ &\textstyle\int_0^t e^{-\int_s^t g(u)du} L_2 g(s)\,ds < 0.063, \end{aligned}$$

$$\int_0^t e^{-\int_s^t g(u)du} (L_3 + L_4) |c(s)| ds \leq 0.125,$$
$$\int_0^t e^{-\int_s^t g(u)du} L_1 |b(s)| ds \leq 0.125.$$

Let $\alpha = 0.063 + 0.189 + 0.189 + 0.121 + 0.063 + 0.125 + 0.125 = 0.875 < 1$, then by Theorem 3.2, every solution $x(t, 0, \psi)$ of (3.14) with small continuous initial function ψ, goes to zero as $t \to \infty$.

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