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ON IRROTATIONAL C-BOCHNER CURVATURE TENSOR IN K-CONTACT AND KENMOTSU MANIFOLDS

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Abstract. The objective of this paper is to study an irrotational C-Bochner curvature tensor in K-contact and Kenmotsu manifolds. It is shown that such manifolds are η-Einstein and examples are also given to verify the results.

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1. Introduction

The authors C.S. Bagewadi and N.B. Gatti ([1], [8]), C.S. Bagewadi, E. Girish Kumar and Venkatesha [2] have studied irrotational projective curvature and quasi-conformal curvature tensors and D-conformal curvature tensor in K-contact, Kenmotsu and trans-Sasakian manifolds and they have shown that these manifolds are Einstein. Further, they have studied some properties like flatness and space of constant curvature.

A K-contact manifold is a differentiable manifold with a contact metric structure such that ξ is a Killing vector field ([5], [13]). These are studied by many authors ([5], [7], [13]). The notion of Kenmotsu manifolds was defined by K. Kenmotsu [9]. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant. For example it is hyperbolic space (-1). Kenmotsu manifolds were studied by many authors such as T.Q. Binh, L. Tamassy, U.C. De, and M. Tarafdar [4], C.S. Bagewadi and Venkatesha [3].

In this paper we study irrotational C-Bochner curvature tensor in K-contact and Kenmotsu manifolds and examples are given to verify the results.

2. Preliminaries

A (2n+1)-dimensional differential manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η on M respectively such that,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{1}$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M, ϕ, ξ, η). If g is a Riemannian metric on an almost contact manifold M such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2}$$

where X, Y are vector fields and it is said to have an almost contact metric structure (ϕ, ξ, η, g) and manifold M equipped with this structure (ϕ, ξ, η, g) is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies,

$$d\eta(X, Y) = g(X, \phi Y), \tag{3}$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and M equipped with a contact metric structure is called contact metric manifold.

If the contact metric structure is normal then it is called a Sasakian structure. Note that an almost contact metric manifold defines Sasakian structure if and only if,

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{4}$$

where ∇ denotes the Riemannian connection on M.

Contact metric manifold with structure tensor (ϕ, ξ, η, g) in which the Killing vector field ξ satisfies

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0, \tag{5}$$

then M is called the K-contact manifold.

An almost contact metric manifold, which satisfies the following conditions,

$$(\nabla_X \phi)Y = \eta(Y)\phi X - g(X, \phi Y)\xi, \tag{6}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{7}$$

is called Kenmotsu manifold.

The C-Bochner curvature tensor [10] is given by

$$\begin{aligned} B(X,Y)Z = {} & R(X,Y)Z + \left(\frac{1}{2n+4}\right)[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX \\ & + S(X,Z)Y + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X \\ & + S(\phi X,Z)\phi Y + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z + \eta(Y)\eta(Z)QX \\ & - \eta(Y)S(X,Z)\xi + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\ & - \frac{D+2n}{2n+4}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ & + \frac{D}{2n+4}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ & - \eta(X)g(Y,Z)\xi] - \frac{D-4}{2n+4}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{8}$$

where $D = \frac{(2n+r)}{(2n+2)}$ and R, S, Q and r are Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively.

The Rotational (curl) of curvature tensor B on a Riemannian manifold is given by

$$\begin{aligned} Rot\,B &= (\nabla_U B)(X,Y)Z + (\nabla_X B)(Y,U)Z + (\nabla_Y B)(U,X)Z \\ &- (\nabla_Z B)(X,Y)U. \end{aligned} \tag{9}$$

Contracting (9) and by virtue of (1), (2) and (8), we have

$$\begin{aligned} Rot\,B &= \frac{2n+3}{2(n+2)}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] + \frac{1}{4(n+1)(n+2)}[n\{g(X,Z)(\nabla_Y r) \\ &- g(Y,Z)(\nabla_X r)\} + \{(\nabla_Y r)\eta(X) - (\nabla_X r)\eta(Y)\}\eta(Z) + (\nabla_\xi r)\{g(X,Z)\eta(Y) \\ &- g(Y,Z)\eta(X)\}] + \frac{2n+r}{4(n+1)(n+2)}[g(X,Z)(\nabla_\xi \eta)(Y) - g(Y,Z)(\nabla_\xi \eta)(X) \\ &+ (2n-2)\{(\nabla_X \eta)(Z)\eta(Y) - (\nabla_Y \eta)(Z)\eta(X) + [(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)]\eta(Z)\}] \\ &- \frac{1}{2(n+2)}[(\nabla_{\phi X} S)(\phi Y,Z) - (\nabla_{\phi Y} S)(\phi X,Z) - 2(\nabla_{\phi Z} S)(\phi X,Y)] \\ &- \frac{2n(2n+3)+r}{4(n+1)(n+2)}[\eta(\nabla_{\phi X}\xi)g(\phi Y,Z) - \eta(\nabla_{\phi Y}\xi)g(\phi X,Z) - 2\eta(\nabla_{\phi Z}\xi)g(\phi X,Y) \\ &- 3\{(\nabla_X \eta)(Z)\eta(Y) + (\nabla_X \eta)(Y)\eta(Z)\} + 3\{(\nabla_Y \eta)(X)\eta(Z) + (\nabla_Y \eta)(Z)\eta(X)\}] \\ &+ \frac{(2n+1)}{4(n+1)(n+2)}[(\nabla_{\phi Y} r)g(\phi X,Z) - (\nabla_{\phi X} r)g(\phi Y,Z) + 2(\nabla_{\phi Z} r)g(\phi X,Y)] \\ &+ \frac{1}{2(n+2)}[\frac{1}{2}\eta(Z)\{(\nabla_X r)\eta(Y) - (\nabla_Y r)\eta(X)\} + r\{(\nabla_Y \eta)(X) - (\nabla_X \eta)(Y)\}\eta(Z) \\ &+ r\{(\nabla_Y \eta)(Z)\eta(X) - (\nabla_X \eta)(Z)\eta(Y)\} + (2n+1)\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ &+ 3\{(\nabla_X S)(\phi Y,\phi Z) + (\nabla_Y S)(\phi X,\phi Z)\} + (\nabla_Y S)(\xi,X)\eta(Z) + (\nabla_X S)(\xi,Y)\eta(Z) \\ &+ (\nabla_\xi S)(Y,Z)\eta(X) - (\nabla_\xi S)(X,Z)\eta(Y) + (\nabla_\xi \eta)(X)S(Y,Z) - (\nabla_\xi \eta)(Y)S(X,Z) \\ &+ (\nabla_X r)g(\phi Y,\phi Z) + (\nabla_X r)g(\phi Y,Z) - (\nabla_Y r)g(\phi X,\phi Z) - (\nabla_Y r)g(\phi X,Z) \\ &+ 2n\{(\nabla_Y S)(\phi X,Z) - (\nabla_X S)(\phi Y,Z) + 2(\nabla_Z S)(\phi X,Y)\} + S(\xi,Y)(\nabla_X \eta)(Z) \\ &+ (\nabla_Y \eta)(Z)S(\xi,X)] + \frac{\nabla_X r}{4(n+1)(n+2)}[-2ng(\phi Y,Z) - 2(n-2)g(\phi Y,\phi Z)] \\ &+ \frac{\nabla_Y r}{4(n+1)(n+2)}[2(n-2)g(\phi X,\phi Z) + 2ng(\phi X,Z)] \\ &- \frac{\nabla_Z r}{(n+1)(n+2)}g(\phi X,Y). \end{aligned} \tag{10}$$

3. Irrotational C-Bochner curvature tensor in K-contact manifold

In this section we show that if $Rot\, B = 0$, then the K-contact manifold is η-Einstein. In a K-contact manifold the following relations hold:

$$\begin{aligned} \nabla_X \xi &= -\phi X, && (11) \\ S(X,\xi) &= 2n\eta(X), && (12) \\ g(R(\xi,X)Y,\xi) &= g(X,Y) - \eta(X)\eta(Y), && (13) \\ R(\xi,X)\xi &= -X + \eta(X)\xi, && (14) \end{aligned}$$

where R and S are the Riemannian curvature tensor and the Ricci tensor of M, respectively. Further, since ξ is a killing vector in K-contact manifold, S and r are invariant under it that is,

$$(L_\xi S) = 0, \qquad (L_\xi r) = 0, \tag{15}$$

where L is Lie derivative.
We know that

$$\begin{aligned} (\nabla_\xi S)(Y,Z) &= \xi S(Y,Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z) \\ &= (L_\xi S)(Y,Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi). \end{aligned} \tag{16}$$

From (11) and (15) in (16), we have

$$(\nabla_\xi S)(Y,Z) = 0, \qquad \nabla_\xi r = 0. \tag{17}$$

Also we know that

$$(\nabla_Y S)(\xi,Z) = YS(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z). \tag{18}$$

Using (12) and (11) in (18), we have

$$\begin{aligned} (\nabla_Y S)(\xi,Z) &= 2nY\eta(Z) - S(Y,\phi Z) - 2n\eta(\nabla_Y Z) \\ &= 2n\{g(\nabla_Y Z,\xi) + g(Z,\nabla_Y \xi)\} - S(Y,\phi Z) - 2n\eta(\nabla_Y Z) \\ &= 2ng(Y,\phi Z) - S(Y,\phi Z). \end{aligned} \tag{19}$$

Let us consider an irrotational C-Bochner curvature tensor in K-contact manifold, that is $Rot\, B = 0$, in (10). In this resulting equation put $X = \xi$ and by virtue of (1), (2), (11) and (12), we get

$$\frac{2n+3}{2(n+2)}[-(\nabla_Y S)(\xi,Z)] + \frac{1}{4(n+2)}[\eta(Z)(\nabla_Y r) - g(Y,Z)(\nabla_\xi r)]$$
$$+\frac{2n+r}{4(n+1)(n+2)}[-(2n-2)(\nabla_Y \eta)(Z)] - \frac{2n(2n+3)+r}{4(n+1)(n+2)}[3(\nabla_Y \eta)(Z)]$$
$$+\frac{1}{2(n+2)}\left[\frac{1}{2}\{(\nabla_\xi r)\eta(Y)\eta(Z) - (\nabla_Y r)\eta(Z)\} + (\nabla_\xi r)\{g(\phi Y,\phi Z) + g(\phi Y, Z)\}\right]$$
$$+\frac{(\nabla_\xi r)}{4(n+1)(n+2)}[-2ng(\phi Y,Z) - 2(n-2)g(\phi Y,\phi Z)] + \frac{1}{2(n+2)}[(\nabla_Y S)(\xi,\xi)\eta(Z)$$
$$-(2n+1)(\nabla_Y S)(\xi,Z) + (\nabla_Y \eta)(Z)S(\xi,\xi) + r(\nabla_Y \eta)(Z)] = 0. \quad (20)$$

By using (17) and (19) in (20), we have

$$\frac{2n+3}{2(n+2)}[-2ng(Y,\phi Z) + S(Y,\phi Z)] + \left[\frac{-2n(8n+7)-(2n+1)r}{4(n+1)(n+2)}\right] g(Y,\phi Z)$$
$$+\frac{1}{2(n+2)}[-4n^2 g(Y,\phi Z) + (2n+1)S(Y,\phi Z) + rg(Y,\phi Z)] = 0. \quad (21)$$

Replace Z by ϕZ in (21), then by (1), we get

$$\frac{2n+3}{2(n+2)}[2ng(Y,Z) - S(Y,Z)] - \left[\frac{-2n(8n+7)-(2n+1)r}{4(n+1)(n+2)}\right] g(\phi Y,\phi Z)$$
$$+\frac{1}{2(n+2)}[4n^2 g(Y,Z) - (2n+1)S(Y,Z)$$
$$+2n\eta(Y)\eta(Z) - rg(\phi Y,\phi Z)] = 0. \quad (22)$$

From (22), we get

$$\begin{aligned} S(Y,Z) &= \left[\frac{n[8n^2+22n+13]}{4(n+1)^2} - \frac{r}{8(n+1)^2}\right] g(Y,Z) \\ &+ \left[-\frac{n(6n+5)}{4(n+1)^2} - \frac{r}{8(n+1)^2}\right] \eta(Y)\eta(Z). \end{aligned} \quad (23)$$

The above relation is of the form $S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z)$, where

$$\alpha = \left[\frac{n[8n^2+22n+13]}{4(n+1)^2} - \frac{r}{8(n+1)^2}\right], \quad \beta = \left[-\frac{n(6n+5)}{4(n+1)^2} - \frac{r}{8(n+1)^2}\right].$$

On contracting (23), we have the scalar curvature r_1, that is

$$r_1 = \frac{n\{(2n+1)(8n^2+22n+13) - (6n+5)\} - r(n+1)}{4(n+1)^2}. \quad (24)$$

Hence we state the following:

Theorem 1. *Let M be a K-contact manifold in which C-Bochner curvature tensor is irrotational then the manifold is η-Einstein and the scalar curvature of such manifold is given in (24).*

4. Irrotational C-Bochner curvature tensor in Kenmotsu manifold

In this section we prove that the Kenmotsu manifold is also η-Einstein, when $Rot\,B = 0$.
In a Kenmotsu manifold M, the following relations hold:

$$S(X,\xi) = -2n\eta(X), \tag{25}$$

$$g(R(\xi,X)Y,\xi) = \eta(X)\eta(Y) - g(X,Y), \tag{26}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{27}$$

for any vector fields X, Y. Further in Kenmotsu manifold we have

$$(L_\xi g) = 2(g - \eta \otimes \eta). \tag{28}$$

For a symmetric endomorphism Q of the tangent space at a point of M, we express the Ricci tensor S as

$$S(X,Y) = g(QX,Y). \tag{29}$$

Using (29) in (28), we have

$$(L_\xi S)(X,Y) = (L_\xi g)(QX,Y) = 2S(X,Y) + 4n\eta(X)\eta(Y). \tag{30}$$

Again by taking (7), (25) and (30) in (16), we get

$$(\nabla_\xi S)(Y,Z) = 0. \tag{31}$$

By using (7) and (25) in (18), we have

$$(\nabla_Y S)(\xi,Z) = -S(Y,Z) - 2ng(Y,Z). \tag{32}$$

Now consider Kenmotsu manifold with $Rot\,B = 0$ in (10). In this resulting equation put $X = \xi$ and by virtue of (1), (2), (7) and (25), we get

$$-\frac{2n+3}{2(n+2)}[(\nabla_Y S)(\xi,Z)]+\frac{1}{4(n+2)}[\eta(Z)(\nabla_Y r)-g(Y,Z)(\nabla_\xi r)]$$
$$+\frac{2n+r}{4(n+1)(n+2)}[-(2n-2)(\nabla_Y\eta)(Z)]-\frac{2n(2n+3)+r}{4(n+1)(n+2)}[3(\nabla_Y\eta)(Z)]$$
$$+\frac{1}{2(n+2)}[\frac{1}{2}\eta(Z)\{(\nabla_\xi r)\eta(Y)-(\nabla_Y r)\}+(\nabla_\xi r)\{g(\phi Y,\phi Z)+g(\phi Y,Z)\}]$$
$$+\frac{(\nabla_\xi r)}{4(n+1)(n+2)}[-2ng(\phi Y,Z)-2(n-2)g(\phi Y,\phi Z)]+\frac{1}{2(n+2)}[(\nabla_Y\eta)(Z)S(\xi,\xi)$$
$$-(2n+1)(\nabla_Y S)(\xi,Z)+(\nabla_Y S)(\xi,\xi)\eta(Z)+r(\nabla_Y\eta)(Z)]=0. \qquad (33)$$

Using (31), (32) in (33), we have

$$\left[\frac{2(n+1)}{(n+2)}\right]S(Y,Z)+\left[\frac{2n[8n^2+6n-1]+r}{4(n+1)(n+2)}\right]g(Y,Z)+\left[\frac{2n(10n+9)-r}{4(n+1)(n+2)}\right]\eta(Y)\eta(Z)$$
$$-\left[\frac{(n-5)}{4(n+1)(n+2)}\right][(\nabla_\xi r)g(\phi Y,\phi Z)]+\left[\frac{1}{2(n+1)(n+2)}\right][(\nabla_\xi r)g(\phi Y,Z)]=0. \quad (34)$$

Interchanging Y and Z in the above equation, we have

$$\left[\frac{2(n+1)}{(n+2)}\right]S(Z,Y)+\left[\frac{2n[8n^2+6n-1]+r}{4(n+1)(n+2)}\right]g(Z,Y)+\left[\frac{2n(10n+9)-r}{4(n+1)(n+2)}\right]\eta(Y)\eta(Z)$$
$$-\left[\frac{(n-5)}{4(n+1)(n+2)}\right][(\nabla_\xi r)g(\phi Z,\phi Y)]+\left[\frac{1}{2(n+1)(n+2)}\right][(\nabla_\xi r)g(\phi Z,Y)]=0. \quad (35)$$

Adding equation (34) and (35), we get

$$S(Y,Z) = \left[\frac{-2n[8n^2+6n-1]-r+(n-5)(\nabla_\xi r)}{8(n+1)^2}\right]g(Y,Z)$$
$$+\left[\frac{-2n(10n+9)+r-(n-5)(\nabla_\xi r)}{8(n+1)^2}\right]\eta(Y)\eta(Z). \qquad (36)$$

the above relation is of the form $S(Y,Z)=\alpha g(Y,Z)+\beta\eta(Y)\eta(Z)$, where

$$\alpha=\left[\frac{-2n[8n^2+6n-1]-r+(n-5)(\nabla_\xi r)}{8(n+1)^2}\right], \quad \beta=\left[\frac{-2n(10n+9)+r-(n-5)(\nabla_\xi r)}{8(n+1)^2}\right].$$

Now contracting (36), we have the scalar curvature r_1, that is

$$r_1=\frac{n[\{2(3n+5)-4n^2(4n+5)\}-r+(n-5)(\nabla_\xi r)]}{4(n+1)^2}. \qquad (37)$$

Hence we state the following:

Theorem 2. *Let M be a Kenmotsu manifold in which C-Bochner curvature tensor is irrotational then the manifold is η-Einstein and the scalar curvature of such manifold is given in (37).*

Remark:

If the scalar curvature r is constant along the characteristic vector ξ that is $\nabla_\xi r = 0$, then the scalar curvature of irrotational Kenmotsu manifold is given by

$$r_1 = \frac{n[\{2(3n+5) - 4n^2(4n+5)\} - r]}{4(n+1)^2}.$$

5.Example

The following examples of contact metric structures 5.1 and 5.2 ([5], [6]) are serve as counter examples to Theorem 1 and Theorem 2:

5.1. Example for K-contact manifold.

Consider the 3-dimensional manifold $C^* \times R$. Let (r, θ, z) be standard coordinates in $C^* \times R$. Let (E_1, E_2, E_3) be linearly independent global frames on $C^* \times R$ given by

$$E_1 = \frac{1}{r}\frac{\partial}{\partial\theta} + r\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial r}, \qquad E_3 = \xi = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0,$$
$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.$$

The (ϕ, ξ, η) is given by

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz - r^2 d\theta,$$
$$\phi E_1 = -E_2, \qquad \phi E_2 = E_1, \qquad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = 1, \qquad \phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W),$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_2] = \frac{1}{r}E_1 - 2E_3, \qquad [E_1, E_3] = [E_2, E_3] = 0.$$

Let ∇ be the Levi-Civita connection with respect to the above metric g given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{38}$$

Thus from Koszula formula we have

$$\begin{aligned} \nabla_{E_1}E_1 &= \frac{-E_2}{r}, \qquad \nabla_{E_2}E_2 = 0, \qquad \nabla_{E_3}E_3 = 0, \\ \nabla_{E_1}E_2 &= \frac{E_1}{r} - E_3, \quad \nabla_{E_2}E_1 = E_3, \quad \nabla_{E_2}E_3 = -E_1, \\ \nabla_{E_1}E_3 &= E_2, \qquad \nabla_{E_3}E_1 = E_2, \qquad \nabla_{E_3}E_2 = -E_1. \end{aligned} \tag{39}$$

The tangent vectors X and Y to $C^* \times R$ are expressed as linear combination of E_1, E_2, E_3, that is $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$, where a_i and b_i are scalars. Clearly (ϕ, ξ, η, g) satisfies the properties of K-contact manifold. Thus $C^* \times R$ is a K-contact manifold. The Ricci tensor $S(X,Y)$ is

$$\begin{aligned} S(X,Y) &= \sum_{i=1}^{3} g(R(X,E_i)E_i, Y) \\ &= g(R(X,E_1)E_1,Y) + g(R(X,E_2)E_2,Y) + g(R(X,E_3)E_3,Y). \end{aligned} \tag{40}$$

The non zero terms $g(R(X,E_i)E_i,Y)$, $i = 1,2,3$ by virtue of (39) are given by

$$\begin{aligned} R(E_2,E_1)E_1 &= -3E_2, \qquad R(E_2,E_3)E_3 = E_2, \\ R(E_3,E_1)E_1 &= E_3, \qquad R(E_1,E_2)E_2 = -3E_1, \\ R(E_3,E_2)E_2 &= E_3, \qquad R(E_1,E_3)E_3 = E_1. \end{aligned} \tag{41}$$

By substituting (41) in (40), we have

$$S(X,Y) = -2g(X,Y) + 4\eta(X)\eta(Y). \tag{42}$$

Now we have to check whether the example satisfies the equation (10) or not:

If $X = Y = Z = E_i$, in (10) and by virtue of (17) and (19), we obtain $Rot\, B = 0$. Thus the Theorem 1 holds true.

However, if the component $(Rot\, B)(E_i,E_i)E_i$ of $(Rot\, B)(X,Y)Z$ where $X \neq Y \neq Z = E_i$, is non zero. Hence in general if $X = \sum_{i=1}^{3} a_i E_i$, $Y = \sum_{i=1}^{3} b_i E_i$, $Z = \sum_{i=1}^{3} c_i E_i$, a_i, b_i, c_i are scalars, then $(Rot\, B)(X,Y)Z \neq 0$. In this case the converse of the Theorem 1 does not hold true.

5.2. Example for Kenmotsu manifold.

We consider 3-dimensional manifold $M = \{(x, y, z) \in R^3; z \neq 0\}$, where (x, y, z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = z\frac{\partial}{\partial x}, \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = -z\frac{\partial}{\partial z}. \tag{43}$$

Let g be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0,$$
$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.$$

The (ϕ, ξ, η) is given by

$$\eta = -\frac{1}{z}dz, \quad \xi = E_3,$$
$$\phi E_1 = -E_2, \quad \phi E_2 = E_1, \quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = 1, \qquad \phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W),$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_2] = 0, [E_1, E_3] = E_1, [E_2, E_3] = E_2.$$

Let ∇ be the Levi-Civita connection with respect to g. From Koszula formula (38), we have

$$\nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_3 = E_2, \quad \nabla_{E_3} E_3 = 0,$$
$$\nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_3} E_2 = 0,$$
$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_3} E_1 = 0. \tag{44}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, that is $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$, a_i, b_i are scalars. Clearly (ϕ, ξ, η, g) is a Kenmotsu structure. Thus M is a Kenmotsu manifold.

The non zero terms $g(R(X, E_i)E_i, Y)$, $i = 1, 2, 3$ by virtue of (44) are given by

$$R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = -E_2,$$
$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = -E_3, \quad R(E_3, E_2)E_2 = -E_3. \tag{45}$$

Now we have to check whether the example satisfies the equation (10) or not:

If $X = Y = Z = E_i$, in (10) and by virtue of (31), (32), we obtain $Rot\, B = 0$. Thus the Theorem 2 holds true.

However, if the component $(Rot\, B)(E_i, E_i)E_i$ of $(Rot\, B)(X, Y)Z$ where $X \neq Y \neq Z = E_i$, is non zero. Hence in general if $X = \sum_{i=1}^{3} a_i E_i$, $Y = \sum_{i=1}^{3} b_i E_i$, $Z = \sum_{i=1}^{3} c_i E_i$, a_i, b_i, c_i are scalars, then $(Rot\, B)(X, Y)Z \neq 0$. In this case the converse of the Theorem 2 does not hold true.

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