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A RELATED FIXED POINT THEOREM FOR N-METRIC SPACES USING CONTINUITY

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Abstract. In this paper we take A_i,i= 1,2,3....n-1 are contonuous mapping from complete metric space X_i to X_{i+1} and A_n is mapping of X_n to X_1,and prove a fixed point theorem on n-metric spaces. This theorem generalize and extends the result obtained by [1],[2],[3] and [4].

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1. Introduction and Preliminaries

In this paper, we are dealing with n-metric space and we prove a fixed point theorem on n-metric space.
Now we give some imortant definition that are used in the continuation of our result.

Definition 1 *Let X be a non-empty set A metric on X in a mapping $d : X \times X \mapsto \mathbb{R}$ such that $\forall\ x, y, z \in X$, the following axioms are satisfied.*

i. $d(x, y) \geqslant 0$

ii. $d(x, y) = 0 \iff x = y$

iii. $d(x, y) = d(y, x)$

iv. $d(x, y) \leqslant d(x, z) + d(z, y)$

The set X together with metric d is called a metric space and we write it as (X, d).

Definition 2 *Let (X, d) be a metric space. A sequence in X_n is a function $x : \mathbb{N} \mapsto X$. The image $x(n)$ of $n \in \mathbb{N}$ is usually denoted by x_n, called the nth term of the sequence $\{x_n\}$.*

Definition 3 *Let (X, d_1) and (Y, d_2) be two metric spaces and let $f : X \mapsto Y$ be a mapping. Then f is said to be continuous at $x \in X$ if and only if the following criterion in satisfied.*

For each $\varepsilon > 0$ $\exists$ a positive number of $\delta(\varepsilon, x)$ such that $d_2(f(x), f(y)) < \varepsilon$ $\forall\ y \in X$ satisfying $d_1(x, y) < \delta$.

If f in continuous at every point of X, then f is called continuous function.

Definition 4 *Let (X, d) be a metric space A sequence $\{x_n\} \in X$ is said to converge to a point $x \in X$ iff the following criterion is satisfied $\forall\ \varepsilon > 0$ $\exists$ a positive integer $m(\varepsilon)$ such that*

$$d(x_n, x) < \varepsilon,\ \forall\ n \geqslant m$$

Definition 5 *A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence iff for each $\varepsilon > 0$ $\exists$ a positive integer $n_0(\varepsilon)$ such that*

$$d(x_m, x_n) < \varepsilon,\ \forall\ n, m \geqslant n_0$$

Definition 6 *A metric space (X, d) is said to be complete iff every Cauchy sequence in X converge to a point of X.*

2. Main Result

Theorem 1 *Let (X_i, d_i) be complete metric spaces where $i = 1, 2, 3, \ldots, n$. If A_i is continuous mapping of X_i to X_{i+1} where $i = 1, 2, 3, \ldots, n-1$ and A_n is mapping of X_n to X_1 satisfying the following inequalities.*

$$\begin{aligned} d_1(A_nA_{n-1}\ldots A_2A_1x^1, A_nA_{n-1}\ldots A_2A_1x_1^1) \\ \leqslant c\max\big\{ & d_1\left(x^1, x_1^1\right), d_1\left(x^1, A_nA_{n-1}\ldots A_2A_1x^1\right), \\ & d_1\left(x_1^1, A_nA_{n-1}\ldots A_2A_1x_1^1\right), d_2\left(A_1x^1, A_1x_1^1\right), \\ & d_3\left(A_2A_1x^1, A_2A_1x_1^1\right), d_4\left(A_3A_2A_1x^1, A_3A_2A_1x_1^1\right), \\ & \ldots, d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1x^1, A_{n-1}A_{n-2}\ldots A_2A_1x_1^1\right)\big\} \end{aligned} \quad (0.1)$$

$$\begin{aligned} d_2(A_1A_n\ldots A_3A_2x^1, A_1A_n\ldots A_3A_2x_1^2) \\ \leqslant c\max\big\{ & d_2\left(x^2, x_1^2\right), d_2\left(x^2, A_1A_n\ldots A_3A_2x^2\right), \\ & d_2\left(x_1^2, A_1A_n\ldots A_3A_2x_1^2\right), d_3\left(A_2x^2, A_2x_1^2\right), \\ & d_4\left(A_3A_2x^2, A_3A_2x_1^2\right), \ldots, \\ & d_n\left(A_{n-1}A_{n-2}\ldots A_3A_2x^2, A_{n-1}A_{n-2}\ldots A_3A_2x_1^2\right), \\ & d_1\left(A_nA_{n-1}\ldots A_3A_2x^2, A_nA_{n-1}\ldots A_3A_2x_1^2\right)\big\} \end{aligned} \quad (0.2)$$

$$d_3(A_2A_1A_n\ldots A_4A_3x^3, A_2A_1A_n\ldots A_4A_3x_1^3)$$

$$\begin{aligned}
&\leqslant c\max\left\{d_3\left(x^3,x_1^3\right),d_3\left(x^3,A_2A_1A_n\ldots A_4A_3x^3\right),\right.\\
&\quad d_3\left(x_1^3,A_2A_1A_n\ldots A_4A_3x_1^3\right),d_4\left(A_3x^3,A_3x_1^3\right),\\
&\quad d_5\left(A_4A_3x^3,A_4A_3x_1^3\right),\ldots,\\
&\quad d_n\left(A_{n-1}A_{n-2}\ldots A_4A_3x^3,A_{n-1}A_{n-2}\ldots A_4A_3x_1^3\right),\\
&\quad d_1\left(A_nA_{n-1}\ldots A_4A_3x^3,A_nA_{n-1}\ldots A_4A_3x_1^3\right)\\
&\quad \left.d_2\left(A_1A_n\ldots A_4A_3x^3,A_1A_n\ldots A_4A_3x_1^3\right)\right\} \qquad (0.3)
\end{aligned}$$

So continuously like above.

$$\begin{aligned}
&d_n(A_{n-1}A_{n-2}\ldots A_1A_nx^n,A_{n-1}A_{n-2}\ldots A_1A_nx_1^n)\\
&\qquad\leqslant c\max\left\{d_n\left(x^n,x_1^n\right),d_n\left(x^n,A_{n-1}A_{n-2}\ldots A_1A_nx^n\right),\right.\\
&\qquad\quad d_n\left(x_1^n,A_{n-1}A_{n-2}\ldots A_1A_nx_1^n\right),d_1\left(A_nx^n,A_nx_1^n\right),\\
&\qquad\quad d_2\left(A_1A_nx^n,A_1A_nx_1^n\right),d_3\left(A_2A_1A_nx^n,A_2A_1A_nx_1^n\right),\\
&\qquad\quad \left.\ldots,d_{n-1}\left(A_{n-2}A_{n-3}\ldots A_1A_nx^n,A_{n-2}A_{n-3}\ldots A_1A_nx_1^n\right)\right\} \qquad (0.4)
\end{aligned}$$

$$\forall\ x',x_1'\in X_1,x^2,x_1^2\in X_2,\ldots,x^n,x_1^n\in X_n$$

where $0\leqslant c<1$. *Then* $A_nA_{n-1}\ldots A_2A_1$ *has a unique fixed point* $\beta_1\in X_1$, $A_1A_n\ldots A_3A_2$ *has a unique fixed point* $\beta_2\in X_2$, $A_2A_1\ldots A_4A_3$ *has a unique fixed point*
$\beta_3\in X_3$ *and so on* $A_{n-1}A_{n-2}\ldots A_1A_n$ *has a unique fixed point* $\beta_n\in X_n$. *Further,*

$$\begin{aligned}
&A_1\left(\beta_1\right)=\beta_2,A_2\left(\beta_2\right)=\beta_3,A_3\left(\beta_3\right)=\beta_4,\\
&\ldots,A_{n-1}\left(\beta_{n-1}\right)=\beta_n,A_n\left(\beta_n\right)=\beta_1
\end{aligned}$$

Proof. Let x_0^1 be an arbitrary point in X_1, let define sequence $\{x_m^1\}$, $\{x_m^2\},\ldots,\{x_m^n\}$ in $X_1,X_2,\ldots,X_n$ respectively by

$$\begin{aligned}
(A_nA_{n-1}\ldots A_3A_2A_1)^mx_0^1&=x_m^1,\\
x_m^2&=A_1\left(x_{m-1}^1\right),\\
x_m^3&=A_2\left(x_m^2\right)\\
&\vdots\\
x_m^n&=A_{n-1}\left(x_m^{n-1}\right)\\
x_m^1&=A_n\left(x_m^n\right)\qquad\text{for } m=1,2,3,\ldots
\end{aligned}$$

Now using inequality (0.2), we have

$$\begin{aligned}
d_2\left(x_m^2, x_{m+1}^2\right) &= d_2\left(A_1 A_n \ldots A_3 A_2 x_{m-1}^2, A_1 A_n \ldots A_3 A_2 x_m^2\right) \\
&\leqslant c \max \left\{d_2\left(x_{m-1}^2, x_m^2\right), d_2\left(x_{m-1}^2, x_m^2\right),\right. \\
&\quad d_2\left(x_m^2, x_{m+1}^2\right), d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right), \\
&\quad \left.\ldots, d_n\left(x_{m-1}^n, x_m^n\right), d_1\left(x_{m-1}^1, x_m^1\right)\right\} \\
d_2\left(x_m^2, x_{m+1}^2\right) &\leqslant c \max \left\{d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_{m-1}^2, x_m^2\right),\right. \\
&\quad d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right), \ldots, \\
&\quad \left.d_n\left(x_{m-1}^n, x_m^n\right)\right\} \qquad (0.5)
\end{aligned}$$

Now by using inequality (0.3), we have,

$$\begin{aligned}
d_3\left(x_m^3, x_{m+1}^3\right) &= d_3\left(A_2 A_1 A_n \ldots A_4 A_3 x_{m-1}^3, A_2 A_1 A_n \ldots A_4 A_3 x_m^3\right) \\
&\leqslant c \max \left\{d_3\left(x_{m-1}^3, x_m^3\right), d_3\left(x_{m-1}^3, x_m^3\right),\right. \\
&\quad d_3\left(x_m^3, x_{m+1}^3\right), d_4\left(x_{m-1}^4, x_m^4\right), d_5\left(x_{m-1}^5, x_m^5\right), \\
&\quad \left.\ldots, d_n\left(x_{m-1}^n, x_m^n\right), d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_m^2, x_{m+1}^2\right)\right\}
\end{aligned}$$

From inequality (0.5) and since $0 \leqslant c < 1$, we get,

$$\begin{aligned}
d_3\left(x_m^3, x_{m+1}^3\right) &\leqslant c \max \left\{d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_{m-1}^2, x_m^2\right),\right. \\
&\quad d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right), \\
&\quad \left.\ldots, d_n\left(x_{m-1}^n, x_m^n\right)\right\} \qquad (0.6)
\end{aligned}$$

Now continuous like above, we get,

$$\begin{aligned}
d_{n-1}\left(x_m^{n-1}, x_{m+1}^{n-1}\right) &\leqslant c \max \left\{d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_{m-1}^2, x_m^2\right),\right. \\
&\quad d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right), \\
&\quad \left.\ldots, d_n\left(x_{m-1}^n, x_m^n\right)\right\} \qquad (0.7)
\end{aligned}$$

Now using inequality (0.4), we get,

$$\begin{aligned}
d_n\left(x_m^n, x_{m+1}^n\right) &= d_n\left(A_{n-1} A_{n-2} \ldots A_1 A_n x_{m-1}^n, A_{n-1} A_{n-2} \ldots A_1 A_n x_m^n\right) \\
&\leqslant c \max \left\{d_n\left(x_{m-1}^n, x_m^n\right), d_n\left(x_{m-1}^n, x_m^n\right),\right. \\
&\quad d_n\left(x_m^n, x_{m+1}^n\right), d_1\left(x_{m-1}^1, x_m^1\right), \\
&\quad d_2\left(x_m^2, x_{m+1}^2\right), d_3\left(x_m^3, x_{m+1}^3\right), \\
&\quad \left.\ldots, d_{n-1}\left(x_m^{n-1}, x_{m+1}^{n-1}\right)\right\}
\end{aligned}$$

From inequalities (0.5), (0.6), (0.7), we get,

$$\begin{aligned} d_n\left(x_m^n, x_{m+1}^n\right) \leqslant c \max \{ & d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_{m-1}^2, x_m^2\right), \\ & d_3\left(x_{m-1}^3, x_m^3\right), \ldots, d_{n-1}\left(x_{m-1}^{n-1}, x_m^{n-1}\right), \\ & d_n\left(x_{m-1}^n, x_m^n\right)\} \end{aligned} \tag{0.8}$$

Now using inequality (0.1), we get,

$$\begin{aligned} d_1\left(x_m^1, x_{m+1}^1\right) &= d_1\left(A_n A_{n-1} \ldots A_2 A_1 x_{m-1}^1, A_n A_{n-1} \ldots A_2 A_1 x_m^1\right) \\ &\leqslant c \max \{ d_1\left(x_{m-1}^1, x_m^1\right), d_1\left(x_{m-1}^1, x_m^1\right), \\ &\quad d_1\left(x_m^1, x_{m+1}^1\right), d_2\left(x_m^2, x_{m+1}^2\right), \\ &\quad d_3\left(x_m^3, x_{m+1}^3\right), \ldots, d_n\left(x_m^n, x_{m+1}^n\right)\} \end{aligned}$$

From inequality (0.7) and (0.8), we get

$$\begin{aligned} d_1\left(x_m^1, x_{m+1}^1\right) \leqslant c \max \{ & d_1\left(x_{m-1}^1, x_m^1\right), d_2\left(x_{m-1}^2, x_m^2\right), \\ & d_3\left(x_{m-1}^3, x_m^3\right), d_4\left(x_{m-1}^4, x_m^4\right), \ldots, \\ & d_n\left(x_{m-1}^n, x_m^n\right)\} \end{aligned} \tag{0.9}$$

It now follows easily by induction on using (0.5), (0.6), (0.7), (0.8) and (0.9).

$$\begin{aligned} d_1\left(x_m^1, x_{m+1}^1\right) &\leqslant c^{m-1} \max \{ d_1\left(x_1^1, x_2^1\right), d_2\left(x_1^2, x_2^2\right) \\ &\qquad d_3\left(x_1^3, x_2^3\right), \ldots, d_n\left(x_1^n, x_2^n\right)\} \\ d_2\left(x_m^2, x_{m+1}^2\right) &\leqslant c^{m-1} \max \{ d_1\left(x_1^1, x_2^1\right), d_2\left(x_1^2, x_2^2\right), \\ &\qquad d_3\left(x_1^3, x_2^3\right), \ldots, d_n\left(x_1^n, x_2^n\right)\} \\ &\;\;\vdots \\ d_{n-1}\left(x_m^{n-1}, x_{m+1}^{n-1}\right) &\leqslant c^{m-1} \max \{ d_1\left(x_1^1, x_2^1\right), d_2\left(x_1^2, x_2^2\right), \\ &\qquad d_3\left(x_1^3, x_2^3\right), \ldots, d_n\left(x_1^n, x_2^n\right)\} \\ d_n\left(x_m^n, x_{m+1}^n\right) &\leqslant c^{m-1} \max \{ d_1\left(x_1^1, x_2^1\right), d_2\left(x_1^2, x_2^2\right), \\ &\qquad d_3\left(x_1^3, x_2^3\right), \ldots, d_n\left(x_1^n, x_2^n\right)\} \end{aligned}$$

Since $0 \leqslant c < 1$ it follows that $\{x_m^1\}$, $\{x_m^2\}$, $\{x_m^3\}, \ldots,$ $\{x_m^{n-1}\}$, $\{x_m^n\}$ are Cauchy sequences with limit β_1, β_2, $\beta_3, \ldots,$ β_{n-1}, $\beta_n \in X_1$, X_2, $X_3, \ldots,$ X_{n-1}, X_n respectively.

Since $A_1, A_2, A_3, \ldots, A_{n-1}$ is continuous mapping so we have,

$$\begin{aligned} \lim_{m\to\infty} x_m^2 &= \lim_{m\to\infty} A_1\left(x_m^1\right) = A_1(\beta_1) = \beta_2 \\ \lim_{m\to\infty} x_m^3 &= \lim_{m\to\infty} A_2\left(x_m^2\right) = A_2(\beta_2) = \beta_3 \\ &\vdots \\ \lim_{m\to\infty} x_m^{n-1} &= \lim_{m\to\infty} A_{n-2}\left(x_m^{n-2}\right) = A_{n-2}(\beta_{n-2}) = \beta_{n-1} \\ \lim_{m\to\infty} x_m^n &= \lim_{m\to\infty} A_{n-1}\left(x_m^{n-1}\right) = A_{n-1}(\beta_{n-1}) = \beta_n \end{aligned} \tag{0.10}$$

Again using inequality (0.1), we have,

$$\begin{aligned} d_1\left(A_nA_{n-1}\ldots A_1\beta_1, x_{m-1}^1\right) &= d_1\left(A_nA_{n-1}\ldots A_1\beta_1, A_nA_{n-1}\ldots A_1x_m^1\right) \\ &\leqslant c\max\left\{d_1\left(\beta_1, x_m^1\right),\right. \\ &\quad d_1\left(\beta_1, A_nA_{n-1}\ldots A_1\beta_1\right), \\ &\quad d_1\left(x_m^1, x_{m+1}^1\right), d_2\left(A_1\beta_1, x_{m+1}^2\right), \\ &\quad d_3\left(A_2A_1\beta_1, x_{m+1}^3\right), \ldots, \\ &\quad \left. d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1, x_{m+1}^n\right)\right\} \end{aligned}$$

Since $A_1, A_2, A_3, \ldots, A_{n-1}$ are continuous. It follows on letting $m \to \infty$ that

$$\begin{aligned} d_1\left(A_nA_{n-1}\ldots A_1\beta_1, \beta_1\right) &\leqslant c\max\{d_1(\beta_1, \beta_1), \\ &\quad d_1\left(\beta_1, A_nA_{n-1}\ldots A_1\beta_1\right), \\ &\quad d_1(\beta_1, \beta_1), d_2(\beta_2, \beta_2), d_3(\beta_3, \beta_3), \\ &\quad \ldots, d_n(\beta_n, \beta_n)\} \\ \Rightarrow \quad d_1\left(A_nA_{n-1}\ldots A_1\beta_1, \beta_1\right) &\leqslant cd_1\left(A_nA_{n-1}\ldots A_2A_1\beta_1, \beta_1\right) \end{aligned}$$

Since $0 \leqslant c < 1$. So, we get,

$$\begin{aligned} & d_1\left(A_nA_{n-1}\ldots A_2A_1\beta_1, \beta_1\right) = 0 \\ \Rightarrow \quad & A_nA_{n-1}\ldots A_2A_1(\beta_1) = \beta_1 \\ \Rightarrow \quad & \beta_1 \text{ is fixed point of } A_nA_{n-1}\ldots A_2A_1 \text{ and } \beta_1 \in X_1 \end{aligned}$$

Similarly,

$$A_1A_n\ldots A_3A_2(\beta_2) = A_1A_n\ldots A_3A_2(A_1\beta_1) = A_1(\beta_1) = \beta_2$$

So β_2 is fixed point of $A_1A_n\ldots A_3A_2$ so on, we get

$$A_{n-1}A_{n-2}\ldots A_1A_n\beta_n = A_{n-1}A_{n-2}\ldots A_1A_nA_{n-1}\beta_{n-1} = A_{n-1}(\beta_{n-1}) = \beta_n$$

So β_n is fixed point of $A_{n-1}A_{n-2}\ldots A_1A_n$. So we hae β_1 is fixed point of $A_nA_{n-1}\ldots A_2A_1$ and $\beta_1 \in X_1$, β_2 is fixed point of $A_1A_n\ldots A_3A_2$ and $\beta_2 \in X_2$, and so on, β_n is fixed point of $A_{n-1}A_{n-2}\ldots A_1A_n$ and $\beta_n \in X_n$.
We now prove that uniqueness of the fixed point β_1. Let suppose β_1' is another fixed point of $A_nA_{n-1}\ldots A_2A_1$ and $\beta_1 \neq \beta_1'$ then using inequality (0.1), we get

$$\begin{aligned} d_1\left(\beta_1,\beta_1'\right) &= d_1\left(A_nA_{n-1}\ldots A_2A_1\beta_1, A_nA_{n-1}\ldots A_2A_1\beta_1'\right) \\ &\leqslant c\max\left\{d_1\left(\beta_1,\beta_1'\right), d_1\left(\beta_1,\beta_1\right), d_1\left(\beta_1',\beta_1'\right)\right. \\ &\quad d_2\left(A_1\beta_1, A_1\beta_1'\right), d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right), \ldots, \\ &\quad \left. d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1, A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right)\right\} \\ d_1\left(\beta_1,\beta_1'\right) &\leqslant c\max\left\{d_2\left(A_1\beta_1, A_1\beta_1'\right), d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right),\right. \\ &\quad \left.\ldots, d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1, A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right)\right\} \end{aligned} \tag{0.11}$$

Now using inequality (0.2), we obtain

$$\begin{aligned} d_2\left(A_1\beta_1, A_1\beta_1'\right) &= d_2\left(A_1A_nA_{n-1}\ldots A_2A_1\beta_1, A_1A_nA_{n-1}\ldots A_2A_1\beta_1'\right) \\ &\leqslant c\max\left\{d_2\left(A_1\beta_1, A_1\beta_1'\right), d_2\left(A_1\beta_1, A_1\beta_1\right),\right. \\ &\quad d_2\left(A_1\beta_1', A_1\beta_1'\right), d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right), \ldots, \\ &\quad d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1, A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right), \\ &\quad \left. d_1\left(\beta_1,\beta_1'\right)\right\} \end{aligned}$$

From inequality (0.11) and since $0 \leqslant c < 1$, we get

$$\begin{aligned} d_2\left(A_1\beta_1, A_1\beta_1'\right) \leqslant c\max&\left\{d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right), d_4\left(A_3A_2A_1\beta_1, A_3A_2A_1\beta_1'\right),\right. \\ &\left.\ldots, d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1,\ A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right)\right\} \end{aligned} \tag{0.12}$$

Now using inequality (0.3). Similarly we get,

$$\begin{aligned} d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right) \leqslant c\max&\left\{d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right), d_3\left(A_2A_1\beta_1, A_2A_1\beta_1\right),\right. \\ & d_3\left(A_2A_1\beta_1', A_2A_1\beta_1'\right), d_4\left(A_3A_2A_1\beta_1, A_3A_2A_1\beta_1'\right), \ldots, \\ & d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1,\ A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right), \\ & \left. d_1\left(\beta_1,\beta_1'\right), d_2\left(A_1\beta_1, A_1\beta_1'\right)\right\} \end{aligned}$$

From inequality (0.11) and (0.12) and since $0 \leqslant c < 1$, we get

$$\begin{aligned} d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right) \leqslant c\max&\left\{d_4\left(A_3A_2A_1\beta_1, A_3A_2A_1\beta_1'\right),\right. \\ & d_5\left(A_4A_3A_2A_1\beta_1, A_4A_3A_2A_1\beta_1\right), \ldots, \\ & d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1,\right. \\ & \left.\left. A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right)\right\} \end{aligned} \tag{0.13}$$

Using inequality (0.4) and similalry we get,

$$d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1, A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right) \leqslant cd_1\left(\beta_1,\beta_1'\right) \qquad (0.14)$$

So by using inequalities (0.11), (0.12), (0.13) and (0.14), we get

$$d_1\left(\beta_1,\beta_1'\right) \leqslant cd_2\left(A_1\beta_1, A_1\beta_1'\right) \leqslant c^2 d_3\left(A_2A_1\beta_1, A_2A_1\beta_1'\right) \leqslant c^3 d_4\left(A_3A_2A_1\beta_1, A_3A_2A_1\beta_1'\right)$$
$$\vdots$$
$$\leqslant c^{n-2} d_n\left(A_{n-1}A_{n-2}\ldots A_2A_1\beta_1,\ A_{n-1}A_{n-2}\ldots A_2A_1\beta_1'\right) \leqslant c^{n-1} d_1\left(\beta_1,\beta_1'\right)$$

So we get,

$$d_1\left(\beta_1,\beta_1'\right) \leqslant c^{n-1} d_1\left(\beta_1,\beta_1'\right)$$

Since $0 \leqslant c < 1$. So, we get,$d_1\left(\beta_1,\beta_1'\right) = 0 \Rightarrow \beta_1 = \beta_1'$ It follows that $\beta_1 = \beta_1'$ and β_1 is the unique fixed point of $A_nA_{n-1}\ldots A_2A_1$. Similarly, we can proved the unicity of other fixed points.

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