Acta Universitatis Apulensis
ISSN: 1582-5329 No. 32/2012
pp. 103-109

ON SOME NEW MAXIMAL AND MINIMAL SETS VIA SEMI-OPEN

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Abstract. The notion of maximal and minimal open sets in a topological space was introduced by [4] and [5]. In this paper, we introduce new classes of sets called maximal semi-open sets and minimal semi-open sets and investigate some of their fundamental properties.

2000 *Mathematics Subject Classification*: Primary: 54A05, 54A10; Secondary: 54E55.

1. Introduction

The study of semi-open sets and their properties were initiated [3], in 1963. Crossley and Hildebrand [2], gave some properties of semi-closure of a set A (denoted by $Cl(A)$), they defined $Cl(A)$ as intersection of all semi-closed sets containing the set A. F. Nakaok and N. Oda [4], and [5], introduced the notation of maximal open sets and minimal open sets in topological spaces. In (2010) M. Caldas et al [1], introduced the notation of maximal θ-open, minimal θ -open, θ-semi maximal open and θ-semi minimal closed and investigate some of the fundamental properties of these sets.
The purpose of the present paper is to introduce the concept of new classes of open sets called maximal semi-open sets and minimal semi-open sets. We also investigate some of their fundamental properties.

2. Preliminaries

Definition 0.1 *[2] A subset A of a space X is said to be semi-open if $A \subset Cl(Int(A))$. The complement of all semi-open set is said to be semi-closed. As the usual sense, the intersection of all semi-closed sets of X containing A is called the semi-closure of A. Also the union of all semi-open sets of X contained A is called the semi-interior of A.*

Definition 0.2 *[6] A subset A of a space X is said to be θ-open set if for each $X \in A$, there exist an open set G such that $X \in G \subset Cl(G) \subset A$.*

Definition 0.3 *A proper nonempty open(resp., θ-open) set U of X is said to be a maximal open [5] (resp.,maximal θ-open)[1] set if any open set (resp.,θ-open) which contains U is X or U.*

Definition 0.4 *[5] A proper nonempty open set U of a space X is said to be a minimal open set if any open set which contained in U is ϕ or U.*

Definition 0.5 *[4] A proper nonempty closed subset F of a space X is said to be a maximal closed set if any closed set which contains F is X or F.*

Definition 0.6 *A proper nonempty closed(resp., θ-closed) subset F of X is said to be a minimal closed [5] (resp.,minimal θ-closed) [1] if any closed set (resp., θ-closed) set which contained in F is ϕ or F.*

3. Maximal and minimal semi-open sets

Definition 0.7 *A proper nonempty semi-open set A of a space X is said to be a maximal semi-open if any semi-open set which contains A is X or A.*

Definition 0.8 *A proper nonempty semi-closed set B of a space X is said to be a minimal semi-closed if any semi-closed set which contained in B is ϕ or B. The family of all maximal semi-open (resp., minimal semi-closed) sets will be denoted by $M_aSO(X)$ (resp., $M_iSC(X)$). We set*
$M_aSO(X,x) = \{A : x \in A \in M_aSO(X)\}$ and
$M_iSC(X,x) = \{F : x \in F \in M_iSC(X)\}$.

Proposition 0.9 *Let A be a proper nonempty subset of X. Then A is a maximal semi-open set if $X \backslash A$ is a minimal semi-closed set.*

Proof. Obvious.

The following example shows that the family of maximal-open sets and maximal semi-open sets are independent in general.

Example 0.10 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$, so the family of semi-open sets is:*
$SO(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then $\{a\}$ is a maximal open set in X, which is not maximal semi-open in X and $\{ a, c\}$ is maximal semi open in X, but not maximal open in X.

Theorem 0.11 *If a maximal semi-open set is open, then it is a maximal open set.*

Proof. Let U be an open and maximal semi-open set in a topological space X. Suppose that U is not maximal open set, then $U \neq X$ and there exists an open set G such that $U \subset G$ and $U \neq G$, but every open set is a semi-open, this implies that G is a semi-open set containing U but $U \neq X$ and $U \neq G$, which is contradiction, Hence U is a maximal open set in X.

Theorem 0.12 *For any topological space X, the following statements are true:*

1. *Let A be a maximal semi-open set and B be semi-open. Then either $A \cup B = X$ or $B \subset A$.*

2. *Let A and B be two maximal semi-open sets. Then either $A \cup B = X$ or $B = A$.*

3. *Let F be a maximal semi-closed set and G be a semi-closed set. Then either $F \cap G = \phi$ or $F \subset G$.*

4. *Let F and G be minimal semi-closed sets. Then either $F \cap G = \phi$ or $F = G$.*

Proof. (1) Let A be a maximal semi-open set and B be a semi-open set. If $A \cup B = X$, then we have done. But if $A \cup B \neq X$, then we have to prove that $B \subset A$. Now $A \cup B \neq X$ means $B \subset A \cup B$ and $A \subset A \cup B$. Therefore we have $A \subset A \cup B$, and A is a maximal semi-open then by Definition0.7, $A \cup B = X$ or $A \cup B = A$, but $A \cup B \neq X$, therefore $A \cup B = A$, which implies that $B \subset A$.
(2) Let A and B be maximal semi-open sets. If $A \cup B = X$, then we have nothing to prove. If $A \cup B \neq X$, then we have to prove that $A = B$. Now $A \cup B \neq X$ means $B \subset A \cup B$ and $A \subset A \cup B$. Now $A \subset A \cup B$ and A is a maximal semi-open, then by Definition 0.7, $A \cup B = X$ or $A \cup B = A$, but $A \cup B \neq X$, therefore $A \cup B = A$, which implies that $B \subset A$. Similarly we obtain that $A \subset B$. Therefore $B = A$
(3) Let F be a minimal semi-closed set and G be a semi-closed set. If $F \cap G = \phi$, then there is nothing to prove. But if $F \cap G \neq \phi$, the we have to prove that $F \subset G$. Now if $F \cap G \neq \phi$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is a minimal semi-closed, then by Definition0.8, $F \cap G = F$ or $F \cap G = \phi$. But $F \cap G \neq \phi$, then $F \cap G = F$ which implies $F \subset G$.

(4) Let F and G be two minimal semi-closed sets. If $F \cap G = \phi$, then there is nothing to prove. But if $F \cap G \neq \phi$, the we have to prove that $F = G$. Now if $F \cap G \neq \phi$,then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is a minimal semi-closed, then by Definition0.8, $F \cap G = F$. But $F \cap G \neq \phi$, then $F \cap G = F$ which implies $F \subset G$. Similarly if $F \cap G \subset G$ and given that G is a minimal semi-closed, then by Definition0.8, $F \cap G = F$ or $F \cap G \neq \phi$. but $F \cap G \neq \phi$, $F \cap G = G$ which implies $G \subset F$. Then $F = G$.

Theorem 0.13 *1. Let A be a maximal semi-open set and x an element of $X \backslash A$. Then $X \backslash A \subset B$ for any semi-closed set B containing x.*

2. Let A be a maximal semi-open set. Then, one of the following is true:
(i) For each $x \in X \backslash A$ and each semi-open set B containing x, $B = X$.
(ii) There exists a semi-open set B such that $X \backslash A \subset B$ and $B \subset X$.

3. Let A be a maximal semi-open set. Then, one of the following is true:
(i) For each $x \in X \backslash A$ and each semi-open set B containing x, we have $X \backslash A \subset B$
(ii) There exists a semi-open set B such that $X \backslash A \subset B \neq X$.

Proof. 1) Since $x \in X \backslash A$, we have B not subset of A for any semi-open set B containing x. Then by Theorem0.11 (1), we have $A \cup B = X$. Therefore $X \backslash A \subset B$.
2) If (i) not true, then there exists an element x of $X \backslash A$ and a semi-open set B containing x such that $B \subset X$. By (1), we have $X \backslash A \subset B$.
3) If (ii) does not true, then we have $X \backslash A \subset B$ for each $x \in X \backslash A$ and each semi-open set B containing x. Hence, we have $X \backslash A \subset B$.

Theorem 0.14 *Let A, B and C be maximal semi-open sets such that $A \neq B$. If $A \cap B \subset C$, then either $A = C$ or $B = C$.*

Proof. Given $A \cap B \subset C$. If $A = C$, then there is nothing to prove. But if $A \neq C$, then using Theorem 0.12 (2), we have
$B \cap C = B \cap [C \cap X] = B \cap [C \cap (A \cup B)]$
$= B \cap [(C \cap A) \cup (C \cap B)]$
$= (B \cap C \cap A) \cup (B \cap C \cap B)$, since $A \cap B \subset C$
$= (A \cup C) \cap B$
$= X \cap B = B$, since $A \cup C = X$. This implies $B \subset C$ and it follows that $B = C$.

Theorem 0.15 *Let A, B and C be maximal semi-open sets which are different from each other. Then $(A \cap B)$ is not a subset of $(A \cap C)$.*

Proof. Let $(A \cap B) \subset (A \cap C)$. Then $(A \cap B) \cup (C \cap B) \subset (A \cap C) \cup (C \cap B)$. Hence $(A \cup C) \cap B \subset C \cap (A \cup B)$. Since by Theorem 0.12 (2), $A \cup C = X$. We have $X \cap B \subset C \cap X$ which implies $B \subset C$. From the definition of maximal semi-open set it follows that $B = C$. Contradiction to the fact A, B and C are different from each other. Therefore $(A \cap B)$ not subset of $(A \cap C)$.

Theorem 0.16 *1. Let F be a minimal semi-closed set of X. If $x \in F$, then $F \subset G$ for any semi closed set G containing x.*

2. Let F be a minimal semi-closed set of X. Then $F = \cap\{G : G \in SC(X)\}$.

Proof. (1) Let $F \in M_iSC(X, x)$ and $G \in SC(X, x)$ such that F not subset of G, this implies that $F \cap G \subset F$ and $F \cap G \neq \Phi$. But F is a minimal semi-closed, by Definition 0.2, $F \cap G = F$ which is contradiction the relation $F \cap G \subset F$. Therefore $F \subset G$.
(2) By (1) and the fact that F is a semi-closed containing x. We have $F \subset \cap\{G : G \in SC(X)\} \subset F$. Therefore we have the result.

Theorem 0.17 *1. Let F and $\{F_\lambda\}_{\lambda\in\Delta}$ be minimal semi-closed sets. If $F \subset \cup_{\lambda\in\Delta}F_\lambda$, then there exists $\lambda \in \Delta$ such that $F = F_\lambda$*

2. Let F and $\{F_\lambda\}_{\lambda\in\Delta}$ be minimal semi-closed sets of X. If $F \neq F_\lambda$ for any $\lambda \in \Delta$, then $\cup_{\lambda\in\Delta}F_\lambda \cap F \neq \phi$.

Proof. (1) Let F and $\{F_\lambda\}_{\lambda\in\Delta}$ be minimal semi-closed sets with $F \subset \cup_{\lambda\in\Delta}F_\lambda$. we have to prove that $F \cap F_\lambda \neq \phi$. Since if $F \cap F_\lambda = \phi$, then $F_\lambda \subset (X\backslash F)$ and hence $F_\lambda \subset \cup_{\lambda\in\Delta}F_\lambda \subset X\backslash F$ which is contradiction. Now as $F \cap F_\lambda \neq \phi$, then $F \cap F_\lambda \subset F$ and $F \cap F_\lambda \subset F_\lambda$, since $F \cap F_\lambda \subset F$ and give that F is a minimal semi-closed, then by Definition 0.2, then $F \cap F_\lambda = F$ or $F \cap F_\lambda = \phi$. But $F \cap F_\lambda \neq \phi$, then $F \cap F_\lambda = F$ which implies that $F \subset F_\lambda$. Similarly $F \cap F_\lambda \subset F_\lambda$ and give that F_λ is minimal semi-closed, then by Definition 0.2, $F \cap F_\lambda = F_\lambda$ or $F \cap F_\lambda = \phi$, but $F \cap F_\lambda \neq \phi$, which implies $F_\lambda \subset F$. Hence $F_\lambda = F$.
(2) Suppose that $(\cup_{\lambda\in\Delta}F_\lambda) \cap F \neq \phi$, then there exists $\lambda \in \Delta$ such that $F_\lambda \cap F \neq \phi$. By Theorem 0.12 (4), we have $F = F_\lambda$ which is contradiction. Hence $(\cup_{\lambda\in\Delta}F_\lambda) \cap F = \phi$.

Theorem 0.18 *Let U be a maximal semi-open set. Then either $sCl(U) = X$ or $sCl(U) = U$.*

Proof. Since U is a maximal semi-open set, then by Theorem 0.13 (2), we have the following cases:
(1) For each $x \in X\backslash U$ and each semi-open set W of x : Let x be any element of

$X \setminus U$ and W be any semi-open set of x. Since $X \setminus U \neq W$, we have $W \cap U \neq \phi$, for any semi-open set W of x. Hence $X \setminus U \subset sCl(U)$. Since $X = X \cup (X \setminus U) \subset U \cup sCl(U) = sCl(U) \subset X$, we have $sCl(U) = X$.
(2) There exists a semi-open set W such that $X \setminus U = W \neq X$: Since $X \setminus U = W$ is a semi-open set, then U is semi-closed set. Therefore, $U = sCl(U)$.

Corollary 0.19 *Let U be a maximal semi-open set. Then either $sInt(X \setminus U) = X \setminus U$ or $sInt(U) = \phi$.*

Proof. Follows from Theorem 0.13.

Theorem 0.20 *Let U be a maximal semi-open set, and S be a nonempty subset of $X \setminus U$. Then $sCl(S) = X \setminus U$.*

Proof. Since $\phi \neq S \subset X \setminus U$, we have $W \cap S \neq \phi$, for any element x of $X \setminus U$ and semi-open set W of x by Theorem 0.18. Then $X \setminus U \subset sCl(U)$. Since $X \setminus U$ is semi-closed set and $S \subset X \setminus U$, we see that $sCl(S) \subset sCl(X \setminus U) = X \setminus U$. Therefore $X \setminus U = sCl(S)$.

Corollary 0.21 *Let U be a maximal semi-open set, and M be a nonempty subset of X with $U \subset M$. Then $sCl(M) = X$.*

Proof. Since $U \subset M \subset X$, there exists a nonempty subset S of $X \setminus U$ such that $M = U \cup S$. Hence, we have $sCL(M) = sCl(U \cup S) = sCl(U) \cup sCl(S) \supset (X \setminus U) \cup U = X$, by Theorem 0.20. Therefore $sCl(M) = X$.

Theorem 0.22 *Let U be a maximal semi-open set, and assume that the subset $X \setminus U$ has at least two elements. Then $sCl(X \setminus \{a\}) = X$, for any element a of $X \setminus U$.*

Proof. Since $U \subset (X \setminus \{a\})$ by our assumption, we have the result by Corollary 0.21.

Theorem 0.23 *Let U be a maximal semi-open set, and N be a proper subset of X with $U \subset N$. Then $sInt(N) = U$.*

Proof. If $N = U$, then $sInt(N) = sInt(U) = U$. Otherwise $N \neq U$, and hence $U \subset N$. It follows that $U \subset sInt(N)$. Since U is maximal semi-open set, we have also $sInt(N) \subset U$. Therefore $sInt(N) = U$.

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