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SOME RESULTS OF $p-$VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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Abstract. In this paper, we derive some properties for $\mathcal{G}_{p,n,l,\delta}(z)$ and $\mathcal{F}_{p,n,l,\delta}(z)$ considering the classes $MT\left(p,\beta_i,\mu_i\right)$, $KD\left(p,\beta_i,\mu_i\right)$ and $N_p\left(\gamma\right)$. Two new subclasses $KD\mathcal{F}_{p,n,l}\left(\beta,\mu,\delta_1,\delta_2,...,\delta_n\right)$ and $KD\mathcal{G}_{p,n,l}\left(\beta,\mu,\delta_1,\delta_2,...,\delta_n\right)$ are defined. Necessary and sufficient conditions for a family of functions f_i and g_i, respectively, to be in the $KD\mathcal{F}_{p,n,l}\left(\beta,\mu,\delta_1,\delta_2,...,\delta_n\right)$ and $KD\mathcal{G}_{p,n,l}\left(\beta,\mu,\delta_1,\delta_2,...,\delta_n\right)$ are defined. As special cases, the properties of $\int\limits_0^z\prod\limits_{i=1}^n\left(f'(t)\right)^\delta dt$ and $\int\limits_0^z\prod\limits_{i=1}^n\left(\frac{f(t)}{t}\right)^\delta dt$ are given.

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1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of the form

$$f(z)=z^p+\sum_{m=p+1}^{\infty}a_m z^m,\qquad \left(p\in\mathbb{N}=\{1,2,...,\}\right), \tag{1}$$

which are analytic in the open disc $\mathcal{U}=\{z\in\mathbb{C}:\ |z|<1\}$. Also denote T_p the subclass of $\mathcal{A}_p$ consisting of functions whose nonzero coefficients, from the second one, are negative and has the form

$$f(z)=z^p-\sum_{m=p+1}^{\infty}a_m z^m,\qquad a_m\geq 0,\quad \left(p\in\mathbb{N}=\{1,2,...,\}\right). \tag{2}$$

Also $\mathcal{A}_1=\mathcal{A}$, $T_1=T$.

A function $f\in\mathcal{A}_p$ is said to be $p-$valently starlike of order $\alpha\,(0\leq\alpha<p)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\qquad (z\in\mathcal{U}).$$

We denote by $\mathcal{S}_p^*(\alpha)$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}_p$ is said to be $p-$valently convex of order $\alpha\,(0 \leq \alpha < p)$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathcal{U})\,.$$

Let $\mathcal{C}_p\,(\alpha)$ denote the class of all those functions which are $p-$valently convex of order α in $\mathcal{U}$.

Note that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the classes of $p-$valently starlike and $p-$valently convex functions in $\mathcal{U}$. Also, we note that $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the usual classes of starlike and convex functions in $\mathcal{U}$.

Let $\mathcal{N}_p\,(\gamma)$ be the subclass of $\mathcal{A}_p$ consisting of the functions f which satisfy the inequality

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < \gamma, \quad (z \in \mathcal{U})\,,\ \gamma > p. \tag{3}$$

Also $\mathcal{N}_1\,(\gamma) = \mathcal{N}\,(\gamma)\,.$ For $p = 1$, this class was studied by Owa (see [12]) and Mohammed (see [9]).

For a function $f \in \mathcal{A}_p$, we define the following operator

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = \frac{1}{p} z f'(z)$$

$$\vdots$$

$$D^k f(z) = D\left(D^{k-1} f(z)\right), \tag{4}$$

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The differential operator D^k was introduced by Shenan et al. (see [18]). When $p = 1$, we get Sălăgean differential operator (see [15]).

We note that if $f \in \mathcal{A}_p$, then

$$D^k f(z) = z^p + \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^k a_m z^m, \quad (p \in \mathbb{N} = \{1, 2, ...\})\,(z \in \mathcal{U})\,.$$

We also note that if $f \in T_p$, then

$$D^k f(z) = z^p - \sum_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^k a_m z^m, \quad (p \in \mathbb{N} = \{1, 2, ...\})\,(z \in \mathcal{U})\,.$$

Let $MT(p,\beta,\mu)$ be the subclass of $\mathcal{A}_p$ consisting of the functions f which satisfy the analytic characterization

$$\left|\frac{z\left(D^{l_i}f(z)\right)'}{D^{l_i}f(z)}-p\right|<\beta\left|\mu\frac{z\left(D^{l_i}f(z)\right)'}{D^{l_i}f(z)}+p\right|, \tag{5}$$

for some $0<\beta\leq p$, $0\leq\mu<p$, $l_i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $z\in\mathcal{U}$. For $p=1$, $l_1=l_2=...=l_n=0$ for all $i=\{1,2,...,n\}$ this class was studied (see [2]).

Definition 1. A function $f\in\mathcal{A}_p$ is said to be in the class $KD(p,\beta,\mu)$ if satisfies the following inequality:

$$\Re\left\{1+\frac{z\left(D^{l_i}f(z)\right)''}{\left(D^{l_i}f(z)\right)'}\right\}\geq\mu\left|1+\frac{z\left(D^{l_i}f(z)\right)''}{\left(D^{l_i}f(z)\right)'}-p\right|+\beta, \tag{6}$$

for some $0\leq\beta<p$, $\mu\geq 0$, $l_i\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $z\in\mathcal{U}$. For $p=1$, $l_1=l_2=...=l_n=0$ for all $i=\{1,2,...,n\}$ this class was studied (see [17], [9]).

Definition 2. Let $l=(l_1,l_2,...,l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$ for all $i=\{1,2,...,n\}$, $n\in\mathbb{N}$. We define the following general integral operators

$$\mathcal{I}_{p,n}^{l,\delta}(f_1,f_2,...,f_n):\mathcal{A}_p^n\to\mathcal{A}_p$$

$$\mathcal{I}_{p,n}^{l,\delta}(f_1,f_2,...,f_n)=\mathcal{F}_{p,n,l,\delta}(z),$$

$$\mathcal{F}_{p,n,l,\delta}(z)=\int_0^z pt^{p-1}\prod_{i=1}^{n}\left(\frac{D^{l_i}f_i(t)}{t^p}\right)^{\delta_i}dt, \tag{7}$$

and

$$\mathcal{J}_{p,n}^{l,\delta}(g_1,g_2,...,g_n):\mathcal{A}_p^n\to\mathcal{A}_p$$

$$\mathcal{J}_{p,n}^{l,\delta}(g_1,g_2,...,g_n)=\mathcal{G}_{p,n,l,\delta}(z),$$

$$\mathcal{G}_{p,n,l,\delta}(z)=\int_0^z pt^{p-1}\prod_{i=1}^{n}\left(\frac{\left(D^{l_i}g_i(t)\right)'}{pt^{p-1}}\right)^{\delta_i}dt, \tag{8}$$

where $f_i,g_i\in\mathcal{A}_p$ for all $i=\{1,2,...,n\}$ and D is defined by (4).

Remark 1. (7) integral operator was studied and introduced by Saltık et al. (see [16]). We note that if $l_1=l_2=...=l_n=0$ for all $i=\{1,2,...,n\}$, then the integral operator $\mathcal{F}_{p,n,l,\delta}(z)$ reduces to the operator $F_p(z)$ which was studied by Frasin (see [6]). Upon setting $p=1$ in the operator (7), we can obtain the integral operator $D^kF(z)$ which was studied by Breaz (see [5]) and Breaz (see [4]). For $p=1$ and $l_1=l_2=...=l_n=0$ in (7), the integral operator $\mathcal{F}_{p,n,l,\delta}(z)$ reduces to the operator $F_n(z)$ which was studied by Breaz, Breaz (see [2]) and Mohammed (see

[10]). Observe that $p = n = 1$, $l_1 = 0$ and $\delta_1 = \delta$, we obtain the integral operator $I_\delta(f)(z)$ which was studied by Pescar and Owa (see [13]), D. Breaz (see [5]) and Mohammed (see [11]) for $\delta_1 = \delta \in [0,1]$ special case of the operator $I_\delta(f)(z)$ was studied by Miller, Mocanu and Reade (see [8]). For $p = n = 1$, $l_1 = 0$ and $\delta_1 = 1$ in (7), we have Alexander integral operator $I(f)(z)$ in (see [1]).

Remark 2. (8) integral operator was studied and introduced by Saltık et al. (see [16]). For $l_1 = l_2 = ... = l_n = 0$ in (8) the integral operator $\mathcal{G}_{p,n,l,\delta}(z)$ reduces to the operator $G_p(z)$ which was studied by Frasin (see [6]). For $p = 1$ and $l_1 = l_2 = ... = l_n = 0$ in (8), the integral operator $\mathcal{G}_{p,n,l,\delta}(z)$ reduces to the operator $G_{\delta_1,\delta_2,...,\delta_n}(z)$ which was studied by Breaz, Breaz and Owa (see [3]) and Mohammed (see [10]). Observe $p = n = 1$, $l_1 = 0$ and $\delta_1 = \delta$, we obtain the integral operator $G(z)$ which was introduced and studied by Pfaltzgraff (see [14]), Mohammed (see [11]), D.Breaz (see [5]) and Kim and Merkes (see [7]).

Now, by using the equations (7) and (8) and the Definition 1 we introduce the following two new subclasses of $KD(p,\beta,\mu)$.

Definition 3. A family of functions $f_i, i = \{1,2,...,n\}$ is said to be in the class $KD\mathcal{F}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ if satisfies the inequality:

$$\Re\left\{1+\frac{z\left(D^{l_i}\mathcal{F}_{p,n,l,\delta}(z)\right)''}{\left(D^{l_i}\mathcal{F}_{p,n,l,\delta}(z)\right)'}\right\} \geq \mu\left|1+\frac{z\left(D^{l_i}\mathcal{F}_{p,n,l,\delta}(z)\right)''}{\left(D^{l_i}\mathcal{F}_{p,n,l,\delta}(z)\right)'}-p\right|+\beta, \tag{9}$$

for some $0 \leq \beta < p$, $\mu \geq 0$, $l_i \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $z \in \mathcal{U}$ where $\mathcal{F}_{p,n,l,\delta}$ is defined in (7).

Definition 4. A family of functions g_i, $i = \{1,2,...,n\}$ is said to be in the class $KD\mathcal{G}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ if satisfies the inequality:

$$\Re\left\{1+\frac{z\left(D^{l_i}\mathcal{G}_{p,n,l,\delta}(z)\right)''}{\left(D^{l_i}\mathcal{G}_{p,n,l,\delta}(z)\right)'}\right\} \geq \mu\left|1+\frac{z\left(D^{l_i}\mathcal{G}_{p,n,l,\delta}(z)\right)''}{\left(D^{l_i}\mathcal{G}_{p,n,l,\delta}(z)\right)'}-p\right|+\beta, \tag{10}$$

for some $0 \leq \beta < p$, $\mu \geq 0$, $l_i \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $z \in \mathcal{U}$ where $\mathcal{G}_{p,n,l,\delta}$ is defined in (8).

2. Sufficient conditions of the operator $F_{p,n,l,\delta}(z)$

First, in this section we prove a sufficient condition for the integral operator $\mathcal{F}_{p,n,l,\delta}(z)$ to be in the class $\mathcal{N}_p(\eta)$.

Theorem 1. Let $l = (l_1,l_2,...,l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1,\delta_2,...,\delta_n) \in \mathbb{R}_+^n$, $0 \leq \mu_i < p$, $0 < \beta_i \leq p$ and $f_i \in \mathcal{A}_p$ for all $i = \{1,2,...,n\}$. If $\left|\frac{\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}\right| < M_i$ and $f_i \in MT(p,\beta_i,\mu_i)$, then the integral operator

$$F_{p,n,l,\delta}(z)=\int_0^z pt^{p-1}\prod_{i=1}^{n}\left(\frac{D^{l_i}f_i(t)}{t^p}\right)^{\delta_i}dt,$$

is in $\mathcal{N}_p(\eta)$, where

$$\eta=\sum_{i=1}^{n}\delta_i\beta_i\left(p+\mu_i M_i\right)+p. \tag{11}$$

Proof. From the definition (7), we observe that $\mathcal{F}_{p,n,l,\delta}(z)\in\mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{F}'_{p,n,l,\delta}(z)=pz^{p-1}\prod_{i=1}^{n}\left(\frac{D^{l_i}f_i(z)}{z^p}\right)^{\delta_i}. \tag{12}$$

Now, we differentiate (12) logarithmically and multiply by z, we obtain

$$1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}=p+\sum_{i=1}^{n}\delta_i\left(\frac{z\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}-p\right).$$

We calculate the real part from both terms of the above expression and obtain

$$\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\}=\sum_{i=1}^{n}\delta_i\Re\left\{\frac{z\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}-p\right\}+p.$$

Since $\Re w\leq |w|\,,then$

$$\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\}\leq\sum_{i=1}^{n}\delta_i\left|\frac{z\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}-p\right|+p.$$

Since $f_i\in MT\left(p,\beta_i,\mu_i\right)$ for all $i=\{1,2,...,n\}$, we have

$$\begin{aligned}\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\} &\leq \sum_{i=1}^{n}\delta_i\beta_i\left|\mu_i\frac{z\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}+p\right|+p, \qquad (13)\\ &\leq \sum_{i=1}^{n}\delta_i\beta_i\mu_i\left|\frac{z\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}\right|+p\sum_{i=1}^{n}\delta_i\beta_i+p.\end{aligned}$$

Since (13) and $\left|\frac{\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}\right|<M_i$, we obtain

$$\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\}<\sum_{i=1}^{n}\delta_i\beta_i\mu_iM_i+p\sum_{i=1}^{n}\delta_i\beta_i+p=\sum_{i=1}^{n}\delta_i\beta_i\left(p+\mu_iM_i\right)+p.$$

Hence $F_{p,n,l,\delta}(z)\in\mathcal{N}_p(\eta)$, $\eta=\sum_{i=1}^{n}\delta_i\beta_i\left(p+\mu_iM_i\right)+p$.

Remark 3. For $p=1$, $l_i=0$ for all $i=\{1,2,...,n\}$ in Theorem 1, we obtain Theorem 1 (see [9]).

Putting $p=1$ in Theorem 1, we have

Corollary 1. Let $l=(l_1,l_2,...,l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n, 0\leq\mu_i<1, 0<\beta_i\leq 1$ and $f_i\in\mathcal{A}$ for all $i=\{1,2,...,n\}$. If $\left|\frac{\left(D^{l_i}f_i\right)'(z)}{D^{l_i}f_i(z)}\right|<M_i$ and $f_i\in MT\left(1,\beta_i,\mu_i\right)$ then the integral operator

$$F_{1,n,l,\delta}(z)=\int\limits_0^z\prod_{i=1}^{n}\left(\frac{D^{l_i}f_i(t)}{t}\right)^{\delta_i}dt,$$

is in $\mathcal{N}(\eta)$, where

$$\eta=\sum_{i=1}^{n}\delta_i\beta_i\left(1+\mu_iM\right)+1.$$

Putting $p=n=1$, $l_1=0$, $\delta_1=\delta$, $\mu_1=\mu$, $\beta_1=\beta$, $M_1=M$ and $f_1=f$ in Theorem 1, we have

Corollary 2. Let $\delta\in\mathbb{R}^+$, $0\leq\mu<1$, $0<\beta\leq 1$ and $f\in\mathcal{A}$. If $\left|\frac{f'(z)}{f(z)}\right|<M$ and $f\in MT\left(1,\beta,\mu\right)$ then the integral operator $\int_0^z\left(\frac{f(t)}{t}\right)^{\delta}$ is in $\mathcal{N}(\eta)$, where $\eta=\delta\beta\left(1+\mu M\right)+1$.

3.Sufficient conditions of the operator $G_{p,n,l,\delta}(z)$

Next, in this section we give a condition for the integral $\mathcal{G}_{p,n,l,\delta}(z)$ to be $p-$ *valently* convex.

Theorem 2. Let $l=(l_1,l_2,...,l_n)\in\mathbb{N}_0^n$, $\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}_+^n$, $\mu_i\geq 0$, $g_i\in KD\left(p,\beta_i,\mu_i\right)$ and let $\beta_i\geq 0$ be real number with the property $0\leq\beta_i<p$ for all $i=\{1,2,...,n\}$. Moreover suppose that $0<\sum_{i=1}^{n}\delta_i\left(p-\beta_i\right)\leq p$, then the integral operator

$$\mathcal{G}_{p,n,l,\delta}(z)=\int_0^z pt^{p-1}\prod_{i=1}^{n}\left(\frac{\left(D^{l_i}g_i(t)\right)'}{pt^{p-1}}\right)^{\delta_i}dt,$$

is convex order of $\sigma = p - \sum_{i=1}^{n} \delta_i (p - \beta_i)$.

Proof. From the definition (8), we observe that $\mathcal{G}_{p,n,l,\delta}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{G}'_{p,n,l,\delta}(z) = pz^{p-1} \prod_{i=1}^{n} \left(\frac{\left(D^{l_i} g_i(z)\right)'}{pz^{p-1}} \right)^{\delta_i}. \tag{14}$$

Now, we differentiate (14) logarithmically and make the similar operators to the proof of the Theorem 2, we obtain

$$1 + \frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)} = p + \sum_{i=1}^{n} \delta_i \left(\frac{z\left(D^{l_i} g_i(z)\right)''}{\left(D^{l_i} g_i(z)\right)'} - p + 1 \right).$$

We calculate the real part from both terms of the above expression and obtain

$$\Re\left\{1 + \frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\} = \sum_{i=1}^{n} \delta_i \Re\left\{1 + \frac{z\left(D^{l_i} g_i(z)\right)''}{\left(D^{l_i} g_i(z)\right)'}\right\} - p\sum_{i=1}^{n} \delta_i + p.$$

Since $g_i \in KD(p, \beta_i, \mu_i)$ for all $i = \{1, 2, ..., n\}$,we have

$$\Re\left\{1 + \frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\} > \sum_{i=1}^{n} \delta_i \left(\mu_i \left| 1 + \frac{z\left(D^{l_i} g_i(z)\right)''}{\left(D^{l_i} g_i(z)\right)'} - p \right| + \beta_i \right) - p\sum_{i=1}^{n} \delta_i + p.$$

Since $\delta_i \mu_i \left| 1 + \frac{z\left(D^{l_i} g_i(z)\right)''}{\left(D^{l_i} g_i(z)\right)'} - p \right| > 0$, we obtain

$$\Re\left\{1 + \frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\} \geq p - \sum_{i=1}^{n} \delta_i (p - \beta_i),$$

which implies that $\mathcal{G}_{p,n,l,\delta}(z)$ is $p-valently$ convex of order $\sigma = p - \sum_{i=1}^{n} \delta_i (p - \beta_i)$.

Remark 4. Setting $p = 1$, $l_i = 0$ and $g_i = f_i$ for all $i = \{1, 2, ..., n\}$ in Theorem 2, we have obtain Theorem 2 in (see [9]).

Putting $p = 1$ in Theorem 2, we have

Corollary 3. Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, $\mu_i \geq 0$, $g_i \in KD(1, \beta_i, \mu_i)$ and let $\beta_i \geq 0$ be real number with the property $0 \leq \beta_i < 1$ for all $i = \{1, 2, ..., n\}$. Moreover suppose that $0 < \sum_{i=1}^{n} \delta_i (1 - \beta_i) \leq 1$, then the integral operator $\mathcal{G}_{1,n,l,\delta}(z) = \int_0^z \prod_{i=1}^{n} \left(\left(D^{l_i} g_i(t)\right)'\right)^{\delta_i} dt$ is convex order of $\sigma = 1 - \sum_{i=1}^{n} \delta_i (1 - \beta_i)$.

Putting $p = n = 1$, $l_1 = 0$, $\delta_1 = \delta$, $\mu_1 = \mu$, $\beta_1 = \beta$ and $g_1 = g$ in Theorem 2, we have

Corollary 4. Let $\delta \in \mathbb{R}^+$, $\mu \geq 0$, $g \in KD(1, \beta, \mu)$ and let $\beta \geq 0$ be real number with the property $0 \leq \beta < 1$. Moreover suppose that $0 < \delta(1-\beta) \leq 1$, then the integral operator $\mathcal{G}_{1,1,0,\delta}(z) = \int_0^z (g'(t))^{\delta} dt$ is convex order of $\sigma = 1 - \delta(1-\beta)$.

4. A necessary and sufficient condition for a family of analytic functions $f_i \in KDF_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$

In this section, we give a necessary and sufficient condition for a family of functions $f_i \in KD\mathcal{F}_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$. Before embarking on the proof of our result, let us calculate the expression $\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}$, required for proving our result.

Recall that, from (7), we have

$$\mathcal{F}'_{p,n,l,\delta}(z) = pz^{p-1} \prod_{i=1}^{n} \left(\frac{D^{l_i} f_i(z)}{z^p} \right)^{\delta_i}. \tag{15}$$

Now, we differentiate (15) logarithmically and multiply by z, we obtain

$$1 + \frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)} - p = \sum_{i=1}^{n} \delta_i \left(\frac{z \left(D^{l_i} f_i\right)'(z)}{D^{l_i} f_i(z)} - p \right).$$

Let $D^{l_i} f_i(z) = z^p - \sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} a_{m,i} z^m$,

$\left(D^{l_i} f_i\right)'(z) = pz^{p-1} - \sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} m a_{m,i} z^{m-1}$ and we get

$$\begin{aligned} 1 + \frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)} - p &= \sum_{i=1}^{n} \delta_i \left[\frac{pz^p - \sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} m a_{m,i} z^m}{z^p - \sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} a_{m,i} z^m} - p \right], \\ &= -\sum_{i=1}^{n} \delta_i \left[\frac{\sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} (m-p) a_{m,i} z^{m-p}}{1 - \sum\limits_{m=p+1}^{\infty} \left(\frac{m}{p}\right)^{l_i} a_{m,i} z^{m-p}} \right]. \end{aligned} \tag{16}$$

Theorem 3. Let the function $f_i \in T_p$ for $i \in \{1, 2, ..., n\}$. Then the functions

$f_i \in KD\mathcal{F}_{p,n,l}\left(\beta, \mu, \delta_1, \delta_2, ..., \delta_n\right)$ for $i \in \{1, 2, ..., n\}$ if and only if

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}(m-p)(\mu+1)a_{m,i}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}}\right] \leq p-\beta. \tag{17}$$

Proof. First consider

$$\mu\left|1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}-p\right|-\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\} \leq (\mu+1)\left|1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}-p\right|.$$

From (16), we obtain

$$\begin{aligned}
& (\mu+1)\left|1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}-p\right|, \\
= \quad & (\mu+1)\left|\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right]\right|, \\
\leq \quad & (\mu+1)\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}(m-p)\left|a_{m,i}\right|\left|z\right|^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}\left|a_{m,i}\right|\left|z\right|^{m-p}}\right], \\
\leq \quad & (\mu+1)\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}}\right].
\end{aligned}$$

If (17) holds then the above expression is bounded by $p-\beta$ and consequently

$$\mu\left|1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}-p\right|-\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\} < -\beta,$$

which equivalent to

$$\Re\left\{1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}\right\} \geq \mu\left|1+\frac{z\mathcal{F}''_{p,n,l,\delta}(z)}{\mathcal{F}'_{p,n,l,\delta}(z)}-p\right|+\beta.$$

Hence $f_i \in KD\mathcal{F}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i \in \{1,2,...,n\}$.

Conversely, let $f_i \in KD\mathcal{F}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i \in \{1,2,...,n\}$ and prove that (17) holds. If $f_i \in KD\mathcal{F}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i \in \{1,2,...,n\}$ and z is real, we get from (7) and (16)

$$\begin{aligned} & p-\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right], \\ \geq\ & \mu\left|\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right]\right|+\beta, \\ \geq\ & \mu\sum_{i=1}^{n}\delta_i\left[\frac{(m-p)\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right]+\beta. \end{aligned}$$

That is equivalent to

$$\begin{aligned} & \sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\mu\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right]+\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right], \\ \leq\ & p-\beta. \end{aligned}$$

The above inequality reduce to

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i(\mu+1)\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}z^{m-p}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^{m-p}}\right]\leq p-\beta.$$

Let $z\to 1^-$ along the real axis, then we get

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i(\mu+1)\left(\frac{m}{p}\right)^{l_i}(m-p)\,a_{m,i}}{1-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}}\right]\leq p-\beta,$$

which give the required result.

Remark 5. Setting $p = 1$, $l_i = 0$ for $i \in \{1, 2, ..., n\}$ in Theorem 3, we have obtain Theorem 3 in (see [9]).

Putting $p = 1$ in Theorem 3, we have

Corollary 5. Let the function $f_i \in T$ for $i \in \{1, 2, ..., n\}$. Then the functions $f_i \in KD\mathcal{F}_{1,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$ for $i \in \{1, 2, ..., n\}$ if and only if

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=2}^{\infty}\delta_i (m)^{l_i}(m-1)(\mu+1)a_{m,i}}{1-\sum\limits_{m=2}^{\infty}(m)^{l_i}a_{m,i}}\right] \leq 1-\beta.$$

Putting $p = n = 1$, $l_1 = 0$, $\delta_1 = \delta$ and $f_1 = f$ in Theorem 3, we have

Corollary 6. Let the function $f \in T$. Then the functions $f \in KD\mathcal{F}_{1,1,0}(\beta, \mu, \delta)$ if and only if

$$\frac{\sum\limits_{m=2}^{\infty}\delta(m-1)(\mu+1)a_{m,1}}{1-\sum\limits_{m=2}^{\infty}a_{m,1}} \leq 1-\beta.$$

5.A necessary and sufffficient condition for a family of analytic functions $g_i \in KDG_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$

In this section, we give a necessary and sufficient condition for a family of functions $g_i \in KD\mathcal{G}_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$. Let us calculate the expression $\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}$, required for proving our result.

Recall that, from (8), we have

$$\mathcal{G}'_{p,n,l,\delta}(z) = pz^{p-1}\prod_{i=1}^{n}\left(\frac{\left(D^{l_i}g_i(z)\right)'}{pz^{p-1}}\right)^{\delta_i}. \tag{18}$$

Now, we differentiate (18) logarithmically and multiply by z, we obtain

$$1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p = \sum_{i=1}^{n}\delta_i\left(\frac{z\left(D^{l_i}g_i(z)\right)''}{\left(D^{l_i}g_i(z)\right)'}-p+1\right).$$

Let $D^{l_i}g_i(z) = z^p - \sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}a_{m,i}z^m$, $\left(D^{l_i}g_i\right)'(z) = pz^{p-1} - \sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-1}$ and

$\left(D^{l_i}g_i\right)''(z) = p(p-1)z^{p-2} - \sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i} m(m-1)a_{m,i}z^{m-2}$, we

$$\begin{aligned}
&1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p \qquad (19)\\
=\ &\sum_{i=1}^{n}\delta_i\left[\frac{p(p-1)z^{p-1}-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m(m-1)a_{m,i}z^{m-1}}{pz^{p-1}-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-1}}-p+1\right],\\
=\ &-\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{n}{p}\right)^{l_i}m(m-p)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right].
\end{aligned}$$

Theorem 4. Let the function $g_i \in T_p$ for $i \in \{1,2,...,n\}$. Then the functions $g_i \in KD\mathcal{G}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i \in \{1,2,...,n\}$ if and only if

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}m(m-p)(\mu+1)a_{m,i}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}}\right]\leq p-\beta. \qquad (20)$$

Proof. First consider

$$\mu\left|1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p\right|-\Re\left\{1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\}\leq(\mu+1)\left|1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p\right|.$$

From (19), we obtain

$$
\begin{aligned}
&(\mu+1)\left|1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p\right| \\
=\ &(\mu+1)\left|\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m\,(m-p)\,a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right]\right|, \\
\leq\ &(\mu+1)\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}m\,(m-p)\,|a_{m,i}|\,|z|^{\,m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m\,|a_{m,i}|\,|z|^{\,m-p}}\right], \\
\leq\ &(\mu+1)\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}m\,(m-p)\,a_{m,i}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}}\right].
\end{aligned}
$$

If (20) holds then the above expression is bounded by $p-\beta$ and consequently

$$\mu\left|1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p\right|-\Re\left\{1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\}<-\beta,$$

which equivalent to

$$\Re\left\{1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}\right\}\geq\mu\left|1+\frac{z\mathcal{G}''_{p,n,l,\delta}(z)}{\mathcal{G}'_{p,n,l,\delta}(z)}-p\right|+\beta.$$

Hence $g_i\in KD\mathcal{G}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i\in\{1,2,...,n\}$.

Conversely, let $g_i\in KD\mathcal{G}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i\in\{1,2,...,n\}$ and prove that (20) holds. If $g_i\in KD\mathcal{G}_{p,n,l}(\beta,\mu,\delta_1,\delta_2,...,\delta_n)$ for $i\in\{1,2,...,n\}$ and z is real, we get from (8) and (19)

$$\begin{aligned} & p-\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right], \\ \geq\ & \mu\left|\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right]\right|+\beta, \\ \geq\ & \mu\sum_{i=1}^{n}\delta_i\left[\frac{\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right]+\beta. \end{aligned}$$

That is equivalent to

$$\begin{aligned} & \sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\mu\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right]+\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right], \\ \leq\ & p-\beta. \end{aligned}$$

The above inequality reduce to

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i\left(\mu+1\right)\left(\frac{m}{p}\right)^{l_i}m\left(m-p\right)a_{m,i}z^{m-p}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}z^{m-p}}\right]\leq p-\beta.$$

Let $z\to 1^-$ along the real axis, then we get $\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=p+1}^{\infty}\delta_i(\mu+1)\left(\frac{m}{p}\right)^{l_i}m(m-p)a_{m,i}}{p-\sum\limits_{m=p+1}^{\infty}\left(\frac{m}{p}\right)^{l_i}ma_{m,i}}\right]\leq$ $p-\beta$,
which give the required result.

Remark 6. Setting $p=1$, $l_i=0$ for $i\in\{1,2,...,n\}$ in Theorem 4, we have obtain Theorem 4 in (see [9]).

Putting $p=1$ in Theorem 4, we have

Corollary 7. Let the function $g_i \in T$ for $i \in \{1, 2, ..., n\}$. Then the functions $g_i \in KD\mathcal{G}_{1,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$ for $i \in \{1, 2, ..., n\}$ if and only if

$$\sum_{i=1}^{n}\left[\frac{\sum\limits_{m=2}^{\infty}\delta_i(\mu+1)(m)^{l_i}m(m-1)a_{m,i}}{1-\sum\limits_{m=2}^{\infty}(m)^{l_i}ma_{m,i}}\right] \leq 1-\beta.$$

Putting $p = n = 1$, $l_1 = 0$, $\delta_1 = \delta$ and $g_1 = g$ in Theorem 4, we have
Corollary 8. Let the function $g \in T$. Then the functions $g \in KD\mathcal{G}_{1,1,0}(\beta, \mu, \delta)$ if and only if $\frac{\sum\limits_{m=2}^{\infty}\delta(\mu+1)m(m-1)a_{m,1}}{1-\sum\limits_{m=2}^{\infty}ma_{m,1}} \leq 1-\beta$.

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