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FUZZY DIFFERENTIAL SUBORDINATION

Georgia Irina Oros and Gheorghe Oros

Abstract. The method of differential subordination or "admissible functions method" was introduced by P.T. Mocanu and S.S. Miller in [1], [2] and then developed in monography [3], as well as in other papers of other mathematicians. Until now, the notion of differential subordination appears in papers published in various journals specialized in complex analysis around the world. We extend this method to the theory of fuzzy sets.

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1. Introduction and preliminaries

In the field of differential equations of real-valued functions often appear examples of differential inequalities that have important applications in the general theory. In those cases, bounds on a function f are often determined from inequalities involving several of the derivatives of f. In two articles [1] and [2], the authors P. T. Mocanu and S. S. Miller extended those ideas involving differential inequalities for real-valued functions to complex valued functions, thus giving birth to a new theory which is known as "the method of differential subordination" or "admissible functions method". This method is one of the newest methods used in geometric theory of analytic functions, having a great merit in getting many new results, as well as simple proofs of known results.

The general form of differential subordination method can be presented as follows:

Let Ω and Δ be any sets in $\mathbb{C}$, let p be an analytic function in the unit disc U with $p(0) = a$, $a \in \mathbb{C}$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. The problem is to study the following implication:

$$\{\psi(p(z), zp'(z), z^2p''(z); z), z \in U\} \subset \Omega \ \text{ implies } \ p(U) \subset \Delta. \tag{1.1}$$

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then there is a conformal mapping g of U onto Δ such that $g(0) = a$. In this case, relation (1.1) can be rewritten as

$$\{\psi(p(z), zp'(z), z^2p''(z); z), z \in U\} \subset \Omega \;\text{ implies }\; p(z) \prec q(z). \tag{1.2}$$

If Ω is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0, 0; 0)$. If in addition, the function $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, then relation (1.1) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \;\text{ implies }\; p(z) \prec q(z),\; z \in U. \tag{1.3}$$

For further details on the differential subordination method, the valuable monograph [3] can be seen.

Let U denote the unit disc of the complex plane $U = \{z \in \mathbb{C} : \; |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C};\; |z| \leq 1\}$ and $\mathcal{H}(U)$ denote the class of analytic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \; f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots,\; z \in U\}$, $\mathcal{A}_n = \{f \in \mathcal{H}(U) : \; f(z) = z + a_{n+1}z^{n+1} + \cdots,\; z \in U\}$, with $\mathcal{A}_1 = \mathcal{A}$. Let $\mathcal{S} = \{f \in \mathcal{A} : f$ univalent in $U\}$ be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0,\; f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development $f(z) = z + a_2z^2 + \cdots,\; z \in U$.

Denote by $S^* = \left\{f \in A;\; \mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0,\; z \in U\right\}$, the class of normalized starlike functions in U, and $K = \left\{f \in A;\; \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0,\; z \in U\right\}$ the class of normalized convex functions in U.

In order to introduce the notion of fuzzy differential subordination, we use the following definitions and lemmas:

Definition 0.1 *[?, p. 21, Definition 2.26]3] We denote by Q the set of functions q that are analytic and injective on $\overline{U} \backslash E(q)$, where $E(q) = \left\{\zeta \in \partial U : \; \lim\limits_{z\to\zeta} q(z) = \infty\right\}$, and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exception set.*

Lemma 0.1 *[?, p. 24, Lemma 2.26]3] Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $p(z) \neq q(z)$ then there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and an $m \geq n > 1$ for which $p(U_{r_0}) \subset q(U)$,*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$ and

(iii) $\mathrm{Re}\, \frac{z_0p''(z_0)}{p'(z_0)} + 1 \geq m\mathrm{Re}\, \left[\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

Definition 0.2 *[4]. Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:*

A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : \ 0 < F_A(x) \le 1\} = \operatorname{supp}(A, F_A)$ is called fuzzy subset. The function F_A is called membership function of the fuzzy set (A, F_A).

Definition 0.3 *[4]. Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*

Proposition 1 *[4] If $(M, F_M) = (N, F_N)$, then we have $M = N$, where $M = \operatorname{supp}(M, F_M)$, $N = \operatorname{supp}(N, F_N)$.*

Proposition 2 *[4] If $(M, F_M) \subseteq (N, F_N)$, then we have $M \subseteq N$, where $M = \operatorname{supp}(M, F_M)$ and $N = \operatorname{supp}(N, F_N)$.*

Definition 0.4 *[4] Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g, written $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$ if the following conditions are satisfied:*

1°. $f(z_0) = g(z_0)$;

2°. $F_{f(D)}(f(z)) \le F_{g(D)}(g(z))$, $z \in D$.

2. Main results

Let $\Omega = \operatorname{supp}(\Omega, F_\Omega) = \{z \in \mathbb{C} \mid 0 < F_\Omega(z) \le 1\}$, $\Delta = \operatorname{supp}(\Delta, F_\Delta) = \{z \in \mathbb{C} \mid 0 < F_\Delta(z) \le 1\}$, $p(U) = \operatorname{supp}(p(U), F_{p(U)}) = \{f(z) \mid 0 < F_{p(U)}(f(z)) \le 1,\ z \in U\}$ and $\psi(\mathbb{C}^3 \times U) = \operatorname{supp}(\psi(\mathbb{C}^3 \times U), F_{\psi(\mathbb{C}^3 \times U)}) = \{\psi(p(z), zp'^2 p''(z); z) \mid 0 < F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'^2 p''(z); z)) \le 1,\ z \in U\}$.

Definition 0.5 *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The **class of admissible functions** $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the **admissibility condition**:*

$$F_\Omega(\psi(r, s, t; z)) = 0, \quad z \in U, \tag{A}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re} \frac{t}{s} + 1 \ge m \operatorname{Re} \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right]$, $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \ge n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$. In the special case when Ω is a simply connected domain, $\Omega \ne \mathbb{C}$, and h is a conformal mapping of U onto Ω we denote this class by $\Psi_n[h, q]$.

If $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$F_\Omega(\psi(q(\zeta), m\zeta q'(\zeta); z)) = 0, \tag{A'}$$

when $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

If $\psi : \mathbb{C} \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$F_\Omega(\psi(q(\zeta); z)) = 0, \tag{A''}$$

when $z \in U$ and $\zeta \in \partial U \setminus E(q)$.

Let (Ω, F_Ω) and (Δ, F_Δ) be any fuzzy sets in $\mathbb{C}$, $(\Omega \subset \mathbb{C},\ \Delta \subset \mathbb{C})$, let p be an analytic function in the unit disc U with $p(0) = a$, $a \in \mathbb{C}$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. The problem is to study the following implication:

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z)) \leq F_\Omega(z), \quad \text{implies} \quad F_{p(U)}(p(z)) \leq F_\Delta(z). \tag{2.1}$$

From this implication we can state the following types of problems that characterize the theory of fuzzy differential subordinations in the complex plane.

Problem 3 *Given (Ω, F_Ω) and (Δ, F_Δ) any fuzzy sets in $\mathbb{C}$, find conditions on ψ so that (2.1) holds. We call such a ψ an admissible function.*

Problem 4 *Given ψ and (Ω, F_Ω), find (Δ, F_Δ) so that (2.1) holds. Furthermore, find the "smallest" such Δ.*

Problem 5 *Given ψ and (Δ, F_Δ), find (Ω, F_Ω) so that (2.1) holds. Furthermore, find the "largest" such (Ω, F_Ω).*

If either (Ω, F_Ω) or (Δ, F_Δ) in (2.1) is a simply connected domain then (2.1) can be rewritten in terms of fuzzy differential subordination.

If (Δ, F_Δ) is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case (2.1) can be rewritten as

$$\begin{gathered} F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z)) \leq F_\Omega(x) \quad \text{implies} \\ F_{p(U)}(p(z)) \leq F_{q(U)}(q(z)),\ z \in U, \quad i.e.\ p(z) \prec_{\mathcal{F}} q(z). \end{gathered} \tag{0.1}$$

If (Ω, F_Ω) is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0, 0; 0)$. If in addition, the function $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, then (2.1) can be rewritten as

$$\begin{gathered} \psi(p(z), zp'(z), z^2p''(z); z) \prec_{\mathcal{F}} h(z) \quad \text{and} \quad F_{\psi(U)}(z) \leq F_{h(U)}(z), \quad \text{implies} \\ p(z) \prec_{\mathcal{F}} q(z) \quad \text{and} \quad F_{p(U)}(z) \leq F_{q(U)}(z), \quad z \in U. \end{gathered} \tag{2.1''}$$

Definition 0.6 *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp'(z), z^2p''(z); z)) \leq F_{h(u)}(h(z)) \quad i.e. \tag{2.2}$$

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_{\mathcal{F}} h(z), \quad z \in U,$$

then p is called a ***fuzzy solution of the fuzzy differential subordination****. The univalent function q is called a* ***fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination****, or more simple a* ***fuzzy dominant****, if $p(z) \prec_{\mathcal{F}} q(z)$, $z \in U$, for all p satisfying (2.2). A fuzzy dominant $\widetilde{q}$ that satisfies $\tilde{q}(z) \prec_{\mathcal{F}} q(z)$, $z \in U$, for all fuzzy dominants q of (2.2) is said to be the fuzzy best dominant of (2.2).*

Note that the fuzzy best dominant is unique up to a rotation of U.

If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)-fuzzy solution, q an (a, n)-fuzzy dominant, and $\tilde{q}$ the best (a, n)-fuzzy solution.

In the case when (Ω, F_Ω) and (Δ, F_Δ) in (2.1) are simply connected domains, we have seen that (2.1) can be rewritten in terms of subordination such as given in (2.2). Using this and Definition 0.5 we can restate Problems 3-5 as follows:

Problem 6 *Given univalent functions h and q, find a class of admissible functions $\Psi[h, q]$ such that (2.2) holds.*

Problem 7 *Given the fuzzy differential subordination in (2.2), find a dominant q. Moreover, find the fuzzy best dominant.*

Problem 8 *Given ψ and fuzzy dominant q, find the "largest" class of univalent function h such that (2.2) holds.*

Next we give solutions to the Problems 3, 6 and 4, 7.

Theorem 9 *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp(z), z^2p''(z); z)) \leq F_\Omega(z), \quad z \in U, \tag{2.3}$$

then $F_{p(U)}(f(z)) \leq F_{q(U)}(g(z))$ i.e. $p(z) \prec_F q(z)$, $z \in U$.

Proof. From (2.3) and Definition 0.3 we have $\psi(p(z), zp'(z), z^2p''(z); z) \subset \Omega$ and this implies

$$F_\Omega\psi(p(z), zp'(z), z^2p''(z); z) \in (0, 1]. \tag{2.4}$$

Assume $p(z) \not\prec_F q(z)$, $z \in U$. By Lemma 0.1 there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and $m \geq n \geq 1$ that satisfy $p(z_0) = q(\zeta_0)$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ and $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re} \left[\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right]$.

Using these conditions with $r = p(z_0) = q(\zeta_0)$, $s = z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, $t = z_0^2 p''(z_0) = \zeta_0^2 q''(\zeta_0)$, and $z = z_0$ in Definition 0.5 we obtain

$$F_\Omega(\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0)) = \psi_\Omega(q(\zeta_0), m\zeta q'(\zeta_0), \zeta_0^2 q''(\zeta_0); z_0) = 0.$$

Since this contradicts (2.4) we must have $p(z) \prec_{\mathcal{F}} q(z)$, $z \in U$. ∎

Remark 2.1 Upon examining the proof of Theorem 9, it is easy to see that the theorem also holds, if condition (2.3) is replaced by

$$F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'(z), z^2 p''(z); w(z)) \leq F_\Omega(z), \quad z \in U,$$

where $w(z)$ is any function mapping U into U.

On checking the definitions of Q and $\Psi_n[\Omega, q]$, we see that the hypothesis of Theorem 9 requires that q behave very nicely on the boundary of U. If this is not true or if the behavior of q is not known, it may still be possible to prove that $p(z) \prec_F q(z)$, by the following limiting procedure.

Theorem 10 *Let $\Omega \subset \mathbb{C}$ and let q be univalent in U, with $q(0) = a$. Let $\psi \in \Psi_n[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$, $z \in U$. If $p \in \mathcal{H}(U)$, with $p(0) = a$ and $F_{\psi(\mathbb{C}^3 \times U)}(\psi(p(z), zp'(z), z^2 p''(z); z)) \leq F_\Omega(z)$, $z \in U$ then $F_{p(U)}(f(z)) \leq F_{q(U)}(q(z))$ i.e. $p(z) \prec_F q(z)$, $z \in U$.*

Proof. The function q_ρ is univalent on $\overline{U}$, and therefore $E(q_\rho)$ is empty and $q_\rho \in Q$. The class $\Psi_n[\Omega, q_\rho]$ is an admissible class and from Theorem 9 we obtain $p(z) \prec_F q_\rho(z)$, $z \in U$.

From $p(z) \prec_F q_\rho(z)$, $z \in U$ we have $p(0) = q_\rho(0)$ and $F_{p(U)}(p(z)) \leq F_{q_\rho(U)}(q_\rho(z))$, $z \in U$. Since $q_\rho(z) = q(\rho z)$, we have $q_\rho(0) = q(0)$ and $F_{q_\rho(U)}(q_\rho(z)) = F_{q(\rho U)}(q(\rho z))$. Hence, $F_{p(U)}(p(z)) \leq F_{q_\rho(U)}(q_\rho(z)) = F_{q(\rho U)}(q(\rho z))$, $p(z) \prec_F q(\rho z)$, $z \in U$.

By letting $\rho \to 1$ we obtain $p(z) \prec_F q(z)$, $z \in U$. ∎

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$ where h is a conformal mapping of U onto Ω and the class $\Psi_n[h(U), q]$ is written as $\psi_n[h, q]$. The following result is an immediate consequence of Theorem 9.

Theorem 11 *Let $\psi \in \Psi_n[h,q]$ with $q(0)=a$. If $p \in \mathcal{H}[a,n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, and $F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp(z), z^2p''(z); z)) \le F_{h(U)}(H(z))$ i.e.*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), \quad z \in U, \quad \textit{then}$$

$$F_{p(U)}(f(z)) \le F_{q(U)}(q(z)) \quad \textit{i.e.} \quad p(z) \prec_F q(z), \quad z \in U.$$

This result can be extended to those cases in which the behavior of q on the boundary of U is unknown by the following theorem.

Theorem 12 *Let h and q be univalent in U, with $q(0)=a$, and let $h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$. Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h, q_\rho]$ for some $\rho \in (0,1)$, or

(ii) there exists $\rho_0 \in (0,1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a,1]$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, and

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp(z), z^2p''(z); z)) \le F_{h(U)}(H(z)) \quad \textit{i.e.} \tag{2.5}$$

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), \quad z \in U, \quad \textit{then}$$

$$F_{p(U)}(f(z)) \le F_{q(U)}(q(z)) \quad \textit{i.e.} \quad p(z) \prec_F q(z), \quad z \in U.$$

Proof. Case (i). By applying Theorem 9, we have $p(z) \prec_F q_\rho(z)$, $z \in U$, i.e. $F_{p(U)}(p(z)) \le F_{q_\rho(U)}(q_\rho(z))$, $z \in U$ and $p(0) = q_\rho(0)$. Since $q_\rho(z) = q(\rho z)$, we have $q_\rho(0) = q(0)$ and $F_{q_\rho(U)}(q_\rho(z)) = F_{q(U)}(q(\rho z))$. We obtain $F_{p(U)}(p(z)) \le F_{q_\rho(U)}(z) = F_{q(U)}(q(\rho z))$, hence $p(z) \prec_F q(\rho z)$, $z \in U$. By letting $\rho \to 1$ we obtain $p(z) \prec_F q(z)$, $z \in U$.

Case (ii). If we let $p_\rho(z) = p(\rho z)$, then $F_{\psi(\mathbb{C}^3\times U)}(\psi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z)) = F_{\psi(\mathbb{C}^3\times U)}(\psi(p(\rho z), \rho z p'^2 z^2 p''(\rho z); \rho z) \le F_{h_\rho(U)}(h_\rho(z))$.

By using Theorem 9 and the comment associated with Remark 2.1, with $w(z) = \rho z$, we obtain $p_\rho(z) \prec_F q_\rho(z)$, for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1$ we obtain $p(z) \prec_F q(z)$, $z \in U$. ■

The next two theorems yield best dominants of the fuzzy differential subordination (2.5). The cases $n=1$ and $n>1$ need to be handled separately.

Theorem 13 *Let h be univalent in U and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(p(z), zp'(z), z^2p''(z); z) = h(z), \quad z \in U, \tag{2.6}$$

has a solution q, with $q(0)=a$, and one of the following conditions is satisfied:

(i) $q \in Q$ *and* $\psi \in \Psi[h, q]$,

(ii) q *is univalent in* U *and* $\psi \in \Psi[h, q_\rho]$, *for some* $\rho \in (0, 1)$, *or*

(iii) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, 1]$ *and* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* U, *and if* p *satisfies*

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp(z), z^2p''(z); z)) \le F_{h(U)}(H(z)) \quad i.e. \tag{2.7}$$

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), \quad then$$

$$F_{p(U)}(f(z)) \le F_{q(U)}(q(z)) \quad i.e. \;\; p(z) \prec_F q(z), \quad z \in U,$$

and q *is the best dominant.*

Proof. By applying Theorems 11 and 12, we deduce that q is a dominant of (2.7). Since q satisfies (2.6), it is a solution of (2.7) and therefore q will be dominated by all dominants of (2.7). Hence q will be the best dominant of (2.7). ■

Theorem 14 *Let* h *be univalent in* U *and let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that the differential equation*

$$\psi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^{2n}q''(z); z) = h(z), \tag{2.8}$$

has a solution q, *with* $q(0) = a$, *and one of the following conditions is satisfied:*

(i) $q \in Q$ *and* $\psi \in \Psi_n[h, q]$;

(ii) q *is univalent in* U *and* $\psi \in \Psi_n[h, q_\rho]$, *for some* $\rho \in (0, 1)$, *or*

(iii) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi_n[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ *is analytic in* U, *and* p *satisfies*

$$F_{\psi(\mathbb{C}^3\times U)}(\psi(p(z), zp(z), z^2p''(z); z)) \le F_{h(U)}(H(z)) \quad i.e. \tag{2.9}$$

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec_F h(z), \quad then$$

$$F_{p(U)}(f(z)) \le F_{q(U)}(q(z)) \quad i.e. \;\; p(z) \prec_F q(z), \quad z \in U,$$

and q *is the best* (a, n)*-dominant.*

Proof. By applying Theorems 11 and 12 we deduce that q is a dominant of (2.9). If we let $p(z) = q(z^n)$, then $zp'(z) = nz^nq'(z^n)$ and $z^2p''(z) = n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n)$.

Therefore from 2.8) we obtain $\psi(p(z), zp'(z), z^2p''(z); z) = h(z^n)$, and using (2.9), we obtain $h(z^n) \prec_F h(z)$, $z \in U$.

Since $p(z) = q(z^n)$, we have $F_{p(U)}(p(z)) = F_{q(U)}(q(z^n))$. Using Definition 0.2 and Proposition 1 we deduce $p(U) = q(U)$ and we conclude that q is the fuzzy (a, n)-best dominant. ■

Remark 2.2 From the above two theorems we see that the problem of finding best dominants of fuzzy differential subordination reduces to finding univalent solutions of differential equations.

Theorem 9 can be used to show that the solutions of certain second order differential equations take values in particular sets.

Corollary 15 *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If f is analytic in U satisfying $f(U) \subset \Omega$ and if the differential equation $\psi(p(z), zp'(z), z^2p''(z); z) = f(z)$ has a solution $p \in \mathcal{H}[a, n]$, then $p(z) \prec_{\mathcal{F}} q(z)$.*

Theorem 16 *Let h be starlike in U, with $h(0) = 0$. If $p \in \mathcal{H}[0, 1] \cap Q$ is univalent in U, then*

$$zp'(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.10}$$

implies $p(z) \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by

$$q(z) = \int_0^z h(t) \cdot t^{-1} dt, \quad z \in U. \tag{2.11}$$

The function q is convex and is the fuzzy best dominant.

Proof. Differentiating (2.11) we have

$$nzq'(z) = h(z), \quad z \in U. \tag{2.12}$$

If we let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(s; z) = s$, for $s = zp'(z)$, $z \in U$, relation (2.10) becomes $\psi(zp'(z)) \prec_{\mathcal{F}} h(z)$, $z \in U$.

Using (2.12) the function q is the solution of the equation $\psi(nzq'(z)) = nzq'(z) = h(z)$, $z \in U$, and since h is starlike, we deduce from Alexander's theorem that q is convex and univalent.

As in the previous theorem, we can assume that h and q are analytic and univalent on $\overline{U}$, and $q'(\zeta) \neq 0$, for $|\zeta| = 1$. The conclusion of this theorem follows from part (i) of Theorem 14 if we show that $\psi \in \Psi_n[h, q]$. We get this immediately since $h(U)$ is a starlike domain and

$$\psi(m\zeta q'(\zeta)) = m\zeta q'(\zeta) = \frac{m}{n} h(\zeta) \notin h(U), \tag{2.13}$$

where $|\zeta| = 1$, $z \in U$ and $m \geq n$. From (2.13) we have $F_{h(U)}(\psi(m\zeta q'(\zeta))) = 0$.

Using Definition 0.5 we obtain $\psi \in \Psi_n[h, q]$. Applying Theorem 14 we conclude that q is the fuzzy best dominant. $\blacksquare$

Example 2.1 Let $h(z) = z + \frac{z^2}{4}\cos z$, $\operatorname{Re} \frac{zh'(z)}{h(z)} > 0$, $z \in U$, with $h(0) = 0$. If $p \in \mathcal{H}[0,1] \cap Q$ satisfies $zp'(z) \prec_{\mathcal{F}} z + \frac{z^2}{4}\cos z = h(z)$, $z \in U$, then $p(z) \prec_{\mathcal{F}} z + \frac{1}{4}z\sin z + \frac{1}{4}\cos z + \frac{1}{4} = q(z)$, $z \in U$, and the function q is convex and is the fuzzy best dominant.

References

[1] Miller, S. S., Mocanu, P. T., *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 298-305.

[2] Miller, S. S., Mocanu, P. T., *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[3] Miller, S. S., Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[4] Oros, G. I., Oros, Gh., *The notion of subordination in fuzzy set theory*, General Mathematics 4(2011), pp.97-103.

Georgia Irina Oros, Gheorghe Oros
Faculty of Sciences
Department of Mathematics
University of Oradea
Str. Universităţii, No.1, 410087 Oradea, Romania
email: *georgia_oros_ro@yahoo.co.uk, gh_oros@yahoo.com*