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CERTAIN CLASSES OF HARMONIC FUNCTIONS ASSOCIATED WITH FOX-WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. The Fox-Wright generalization of the classical hypergeometric function is used to introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. Among the results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions.

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1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

We denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \;\; g(z) = \sum_{m=1}^{\infty} b_m z^m \;\; (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (|b_1| < 1). \tag{2}$$

It may be noted that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

Also, we denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m + \overline{\sum_{m=1}^{\infty} |b_m| z^m} \ \ (|b_1| < 1). \tag{3}$$

The Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m \tag{4}$$

and

$$\varphi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m \tag{5}$$

is defined (as usual) by $(\phi * \varphi)(z) = \phi(z) * \varphi(z) = z + \sum\limits_{m=2}^{\infty} \lambda_m \mu_m z^m$.

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_p, A_p$ and $\beta_1, B_1, \ldots, \beta_q, B_q (p, q \in N_0 = N \cup \{0\})$ satisfying the condition that $1 + \sum\limits_{m=1}^{q} B_m - \sum\limits_{m=1}^{p} A_m \geq 0 \ \ (z \in U)$, the Fox-Wright generalization

$${}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] ={}_p \Psi_q[(\alpha_m, A_m)_{1,p} (\beta_m, B_m)_{1,q}; z]$$

of the hypergeometric function ${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$ is defined by [12]; (see also [10].)

$${}_p\Psi_q[(\alpha_m, A_m)_{1,p} (\beta_m, B_m)_{1,q}; z] = \sum_{m=0}^{\infty} \{\prod_{n=1}^{p} \Gamma(\alpha_n + mA_n)\}\{\prod_{n=1}^{q} \Gamma(\beta_n + mB_n\}^{-1} \frac{z^m}{m!} (z \in U).$$

If $A_n = 1 (n = 1, ..., p)$ and $B_n = 1 (n = 1, ..., q)$, then we have the following obvious relationship:

$$\begin{aligned} \Theta_p \Psi_q[(\alpha_n, 1)_{1,p} (\beta_n, 1)_{1,q}; z] \ &\equiv \ {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z) \\ &= \ \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \ \frac{z^m}{m!} \end{aligned} \tag{6}$$

where ${}_pF_q(\alpha_1,\dots\alpha_p;\beta_1,\dots,\beta_q;z)$ is the generalized hypergeometric function (see for details [4]), $(\alpha)_m=\alpha(\alpha+1)\dots(\alpha+m-1)$ is the familiar Pochhammer symbol, and Θ is given by

$$\Theta=\left(\prod_{n=0}^{p}\Gamma(\alpha_n)\right)^{-1}\left(\prod_{n=0}^{q}\Gamma(\beta_n)\right). \tag{7}$$

By using the generalized hypergeometric function, Dziok and Srivastava [4] introduced a linear operator which was subsequently extended by Dziok and Raina [3] by using the Fox-Wright generalized hypergeometric function .

Let $W[(\alpha_n,A_n)_{1,p};(\beta_n,B_n)_{1,q}]:S\rightarrow S$ be a linear operator defined by

$$W[(\alpha_n,A_n)_{1,p};(\beta_n,B_n)_{1,q}]\phi(z) \quad := \quad \{\Theta z\ {}_p\Psi_q[(\alpha_n,A_n)_{1,p};(\beta_n,B_n)_{1,q};z]\}*\phi(z),$$

then on using (4) and (7), we get

$$W[(\alpha_n,A_n)_{1,p};(\beta_n,B_n)_{1,q}]\phi(z)=z+\sum_{m=2}^{\infty}\Theta\sigma_m(\alpha_1)\ \lambda_m z^m, \tag{8}$$

where Θ is defined by (7), and $\sigma_m(\alpha_1)$ is given by

$$\sigma_m(\alpha_1)=\frac{\Theta\Gamma(\alpha_1+A_1(m-1))\dots\Gamma(\alpha_p+A_p(m-1))}{(m-1)!\Gamma(\beta_1+B_1(m-1))\dots\Gamma(\beta_q+B_q(m-1))}\ . \tag{9}$$

For convenience sake, we adopt the contracted notation $W_q^p[\alpha_1]$ to represent the following:

$$W_q^p[\alpha_1]\phi(z)=W[(\alpha_1,A_1),\dots,(\alpha_l,A_p);(\beta_1,B_1),\dots,(\beta_q,B_q)]\phi(z), \tag{10}$$

which we use in the sequel throughout. The linear operator $W_q^p[\alpha_1]$ contains the Dziok-Srivastava operator (see [4]), and as its various special cases contain such linear operators as the Hohlov operator, Carlson-Shaffer operator, Ruscheweyh derivative operator, generalized Bernardi-Libera-Livingston operator and fractional derivative operator. Details and references about these operators can be found in [3] and [4].

In view of the relationship (6) and the linear operator (8) for the harmonic function $f=h+\overline{g}$ given by (1), we define the operator

$$W_q^p[\alpha_1]f(z)=W_q^p[\alpha_1]h(z)+\overline{W_q^p[\alpha_1]g(z)}, \tag{11}$$

and introduce below a new subclass $W_H([\alpha_1],\lambda,\gamma)$ of $\mathcal{H}$ in terms of the operator defined by (11) .

Goodman [5] introduced two interesting subclasses of S. One of the subclasses of S is the class UCV of uniformly convex functions and its analytic characterization is defined as follows: A function $\phi \in UCV$ if and only if

$$\mathrm{Re}\left\{1+\frac{z\phi''(z)}{z\phi'(z)}\right\} \geq \mathrm{Re}\left\{\frac{\zeta\phi''(z)}{z\phi'(z)}\right\}, (z,\zeta)\in\mathcal{U}\times\mathcal{U}.$$

Upon choosing $\zeta = -e^{i\psi}z$, the above assertion becomes that $\phi \in UCV$ if and only if Re $\left\{1+(1+e^{i\psi})\frac{z\phi''(z)}{z\phi'(z)}\right\} \geq 0$, where ψ is real.

In order to consider extension of the class UCV to include the harmonic functions, we introduce here a new subclass $W_q^pG_{\mathcal{H}}([\alpha_1],\gamma)$ of $\mathcal{H}$ consisting of harmonic functions $f \in \mathcal{H}$ of the form (1) such that

$$\mathrm{Re}\ \left\{1+(1+e^{i\psi})\frac{z^2(W_q^p[\alpha_1]h(z))''+\overline{2z(W_q^p[\alpha_1]g(z))'+z^2(W_q^p[\alpha_1]g(z))''}}{z(W_q^p[\alpha_1]h(z))'-\overline{z(W_q^p[\alpha_1]g(z))'}}\right\} \geq \gamma, \tag{12}$$

where $0 \leq \gamma < 1$ $(z \in \mathcal{U})$ and $W_q^p[\alpha_1]f(z)$ is defined by (11).

We also let $WT_{\mathcal{H}}([\alpha_1],\gamma) = W_q^pG_{\mathcal{H}}([\alpha_1],\gamma)\bigcap T_{\mathcal{H}}$, where $T_{\mathcal{H}}$ ([9]) is the class of harmonic functions f such that

$$f(z) = z - \sum_{m=2}^{\infty}|a_m|z^m - \overline{\sum_{m=1}^{\infty}|b_m|z^m}, \ \ |b_1|<1. \tag{13}$$

We deem it appropriate to mention here some of the useful subclasses which stem from the class $W_q^pG_{\mathcal{H}}([\alpha_1],\gamma)$ defined above by (12). Indeed, for suitable choices of p, q and other involved parameters, the family $W_q^pG_{\mathcal{H}}([\alpha_1],\gamma)$ from (12) reduces to the classes which we illustrate below.

(i) If we put $A_n = 1 (n = 1, ..., p)$ and $B_n = 1 (n = 1, ..., q)$, then the family $W_q^pG_{\mathcal{H}}([\alpha_1],\gamma)$ defined by (12) reduces to the class denoted by $HG_{\mathcal{H}}([\alpha_1],\gamma)$ which satisfies the inequality:

$$\mathrm{Re}\ \left\{1+(1+e^{i\psi})\frac{z^2(H_q^p([\alpha_1]h(z))''+\overline{2z(H_q^p([\alpha_1]g(z))'+z^2(H_q^p([\alpha_1]g(z))''}}{z(H_q^p([\alpha_1]h(z))'-\overline{z(H_q^p([\alpha_1]g(z))'}}\right\} \geq \gamma$$

where $H_q^p([\alpha_1])$ is the Dziok - Srivastava operator [4].

(ii) Next, in view of the relationship $W_1^2([a,1;c]) = \mathcal{L}(a,c)f(z)$, we obtain a class $LG_{\mathcal{H}}([\alpha_1],\gamma)$ satisfying the inequality:

$$\mathrm{Re}\ \left\{1+(1+e^{i\psi})\frac{z^2(\mathcal{L}(a,c)h(z))''+\overline{2z(\mathcal{L}(a,c)g(z))'+z^2(\mathcal{L}(a,c)g(z))''}}{z(\mathcal{L}(a,c)h(z))'-\overline{z(\mathcal{L}(a,c)g(z))'}}\right\} \geq \gamma,$$

where $\mathcal{L}(a,c)$ is the Carlson - Shaffer operator [1].

(iii) Also, by noting the relationship $W_1^2([\lambda+1,1;1]) = D^\delta f(z)$, we arrive at the class $RG_{\mathcal{H}}([\alpha_1],\gamma)$ which satisfies the inequality:

$$\text{Re}\ \left\{1+(1+e^{i\psi})\frac{z^2(D^\delta h(z))'' + \overline{2z(D^\delta g(z))' + z^2(D^\delta g(z))''}}{z(D^\delta h(z))' - \overline{z(D^\delta g(z))'}}\right\} \geq \gamma,$$

where $D^\delta f(z)(\delta > -1)$ is the Ruscheweyh derivative operator [8] (also see [7]).

(iv) Lastly, in view of the relationship $W_1^2([2,1;2-\mu]) = \Omega_z^\mu f(z)$, we obtain another class $FG_{\mathcal{H}}([\alpha_1],\gamma)$ satisfying the condition that

$$\text{Re}\ \left\{1+(1+e^{i\psi})\frac{z^2(\Omega_z^\mu h(z))'' + \overline{2z(\Omega_z^\mu g(z))' + z^2(\Omega_z^\mu g(z))''}}{z(\Omega_z^\mu h(z))' - \overline{z(\Omega_z^\mu g(z))'}}\right\} \geq \gamma,$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator [11] given by

$$\Omega_z^\mu f(z) = \Gamma(2-\mu) z^\mu D_z^\mu f(z)(0 \leq \mu < 1).$$

Motivated by the earlier works of [6,7,9] on the subject of harmonic functions, we in this paper obtain first a sufficient coefficient condition for function f given by (2) to be in the class $W_q^p G_{\mathcal{H}}([\alpha_1],\gamma)$. It is then shown that this coefficient condition is necessary also for functions belonging to the class $WT_{\mathcal{H}}([\alpha_1],\gamma)$. Further, distortion results and extreme points for functions in $WT_{\mathcal{H}}([\alpha_1],\gamma)$ are also obtained.

2. The class $W_q^p G_{\mathcal{H}}([\alpha_1],\gamma)$.

We begin by stating and proving a sufficient coefficient condition for the function of the form (2) to belong to the class $W_q^p G_{\mathcal{H}}([\alpha_1],\gamma)$. This result is contained in the following.

Theorem 1. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{m=1}^{\infty} m\left(\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\right)\sigma_m(\alpha_1) \leq 2, \tag{14}$$

where $0 \leq \gamma < 1$, then $f \in W_q^p G_{\mathcal{H}}([\alpha_1],\gamma)$.

Proof. We first show that if the inequality (14) holds for the coefficients of $f = h+\overline{g}$, then the required condition (12) is satisfied. Using (11) and (12), we can write

$$\text{Re}\ \left\{\frac{z(W_q^p[\alpha_1]h(z))' + (1+e^{i\psi})z^2(W_q^p[\alpha_1]h(z))'' + (1+2e^{i\psi})\overline{z(W_q^p[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(W_q^p[\alpha_1]g(z))''}}{z(W_q^p[\alpha_1]h(z))' - \overline{z(W_q^p[\alpha_1]g(z))'}}\right\}$$

$$= \text{Re}\ \frac{A(z)}{B(z)}$$

where $A(z) = z(W_q^p[\alpha_1]h(z))' + (1+e^{i\psi})z^2(W_q^p[\alpha_1]h(z))'' + (1+2e^{i\psi})\overline{z(W_q^p[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(W_q^p[\alpha_1]g(z))''}$ and $B(z) = z(W_q^p[\alpha_1]h(z))' - \overline{z(W_q^p[\alpha_1]g(z))'}$. In view of the simple assertion that Re $(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{15}$$

Substituting the above appropriate expressions for $A(z)$ and $B(z)$ in (15), we get

$$\begin{aligned}
& |A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \\
\geq \; & (2-\gamma)|z| - \sum_{m=2}^{\infty} m(2m-\gamma)\sigma_m(\alpha_1)|a_m|\;|z|^m - \sum_{m=1}^{\infty} m(2m+\gamma)\sigma_m(\alpha_1)|b_m|\;|\;|z|^m \\
& -\gamma|z| - \sum_{m=2}^{\infty} m(2m-2-\gamma)\sigma_m(\alpha_1)|a_m|\;|z|^m \\
& - \sum_{m=1}^{\infty} m(2m+2+\gamma)\sigma_m(\alpha_1)|b_m|\;|z|^m. \\
\geq \; & 2(1-\gamma)|z| \left\{ 1 - \sum_{m=2}^{\infty} m\frac{2m-1-\gamma}{1-\gamma}\sigma_m(\alpha_1)|a_m| - \sum_{m=1}^{\infty} m\frac{2m+1+\gamma}{1-\gamma}\sigma_m(\alpha_1)|b_m| \right\} \\
\geq \; & 0
\end{aligned}$$

by virtue of the inequality (14). This implies that $f \in G_{\mathcal{H}}([\alpha_1], \gamma)$.

The following result gives a necessary and sufficient condition for the function given by (13) to belong to $WT_{\mathcal{H}}([\alpha_1], \gamma)$.

Theorem 2. *Let $f = h + \overline{g}$ be given by (13). Then $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$ if and only if*

$$\sum_{m=1}^{\infty} m \left\{ \frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m| \right\} \sigma_m(\alpha_1) \leq 2, \tag{16}$$

where $0 \leq \gamma < 1$.

Proof. Since $WT_{\mathcal{H}}([\alpha_1], \gamma) \subset W_q^p G_{\mathcal{H}}([\alpha_1], \gamma)$, we only need to prove the necessary part of the theorem. Assume that $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$, then by virtue of (11) to (12), we obtain

$$\text{Re } \left\{ (1-\gamma) + (1+e^{i\psi}) \frac{z^2 W_q^p[\alpha_1]h(z))'' + \overline{2zW_q^p[\alpha_1]g(z))' + z^2 W_q^p[\alpha_1]g(z))''}}{zW_q^p[\alpha_1]h(z))' - \overline{zW_q^p[\alpha_1]g(z))'}} \right\} \geq 0.$$

The above inequality is equivalent to

$$\text{Re}\left\{\frac{z-\left(\sum\limits_{m=2}^{\infty} m[m(1+e^{i\psi})-\gamma-e^{i\psi}]\sigma_m(\alpha_1)|a_m|z^m+\sum\limits_{m=1}^{\infty} m[m(1+e^{i\psi})+\gamma+e^{i\psi}]\sigma_m(\alpha_1)|b_m|\overline{z}^m\right)}{z-\sum\limits_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|z^m+\sum\limits_{m=2}^{\infty}\sigma_m(\alpha_1)|b_m|\overline{z}^m}\right\}$$

$$= \text{Re}\left\{\frac{(1-\gamma)-\sum\limits_{m=2}^{\infty} m[m(1+e^{i\psi})-e^{i\psi}-\gamma]\sigma_m(\alpha_1)|a_m|z^{m-1}}{1-\sum\limits_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}\sigma_m(\alpha_1)|b_m|\overline{z}^{m-1}}\right\}$$

$$- \text{Re}\left\{\frac{\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty} m[m(1+e^{i\psi})+e^{i\psi}+\gamma]\sigma_m(\alpha_1)|b_m|\overline{z}^{m-1}}{1-\sum\limits_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum\limits_{m=1}^{\infty}\sigma_m(\alpha_1)|b_m|\overline{z}^{m-1}}\right\}\geq 0.$$

This condition must hold for all values of z such that $|z| = r < 1$. Upon noting that $Re(-e^{i\psi}) \geq -|e^{i\psi}| = -1$, the above inequality reduces to

$$\frac{(1-\gamma)-\left[\sum\limits_{m=2}^{\infty} m(2m-1-\gamma)\sigma_m(\alpha_1)|a_m|r^{m-1}+\sum\limits_{m=1}^{\infty} m(2m+1+\gamma)\sigma_m(\alpha_1)|b_m|r^{m-1}\right]}{1-\sum\limits_{m=2}^{\infty}\sigma_m(\alpha_1)|a_m|r^{m-1}+\sum\limits_{m=1}^{\infty}\sigma_m(\alpha_1)|b_m|r^{m-1}}$$

$$\geq 0. \tag{17}$$

If (16) does not hold, then the numerator in (17) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in (0,1) for which the quotient in (17) is negative. This contradicts our assumption that $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (16) holds true when $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$. This completes the proof of Theorem 2.

3.Distortion and Extreme Points

In this section we obtain the distortion bounds for the functions $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$ that lead to a covering result for the family $WT_{\mathcal{H}}([\alpha_1], \gamma)$.

Theorem 3.*If $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$ then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right)r^2.$$

Proof. We will only prove the right- hand inequality of the above theorem. The arguments for the left- hand inequality are similar and so we omit it. Let $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$ taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \\ &\leq (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|). \end{aligned}$$

This implies that

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r \\ &\quad + \frac{1}{2\sigma_2(\alpha_1)}\frac{1-\gamma}{(3-\gamma)}\left(\sum_{m=2}^{\infty}\frac{3-\gamma}{1-\gamma}2\sigma_2(\alpha_1)|a_m| + \frac{3-\gamma}{1-\gamma}2\sigma_2(\alpha_1)|b_m|\right)r^2 \\ &\leq (1+|b_1|)r + \frac{1}{2\sigma_2(\alpha_1)}\frac{1-\gamma}{(3-\gamma)}\left[1-\frac{3+\gamma}{1-\gamma}|b_1|\right]r^2 \\ &\leq (1+|b_1|)r + \frac{1}{2\sigma_2(\alpha_1)}\left(\frac{1-\gamma}{3-\gamma}-\frac{3+\gamma}{3-\gamma}|b_1|\right)r^2, \end{aligned}$$

which establishes the desired inequality.
As a consequence of the above theorem , we state the following covering lemma.

Corollary 1.*Let $f = h + \overline{g}$ and of the form (2) be so that $f \in WT_{\mathcal{H}}([\alpha_1], \gamma)$. Then*

$$\left\{w : |w| < \frac{6\sigma_2(\alpha_1) - 1 - [2\sigma_2(\alpha_1)-1]\gamma}{2(3-\gamma)\sigma_2(\alpha_1)} - \frac{6\sigma_2(\alpha_1) - 1 - [2\sigma_2(\alpha_1)+1]\gamma}{2(3-\gamma)\sigma_2(\alpha_1)}b_1\right\} \subset f(\mathcal{U}).$$

Next we determine the extreme points of closed convex hulls of $WT_{\mathcal{H}}([\alpha_1]\gamma)$ denoted by $clcoWT_{\mathcal{H}}([\alpha_1], \gamma)$.
Theorem 4. *A function $f(z) \in WT_{\mathcal{H}}([\alpha_1], \gamma)$ if and only if*

$$f(z) = \sum_{m=1}^{\infty}\left(X_m h_m(z) + Y_m g_m(z)\right)$$

where

$$h_1(z) = z, h_m(z) = z - \frac{1-\gamma}{m(2m-1-\gamma)\sigma_m(\alpha_1)} z^m; \quad (m \geq 2),$$
$$g_m(z) = z - \frac{1-\gamma}{m(2m+1+\gamma)\sigma_m(\alpha_1)} \overline{z}^m; \quad (m \geq 2),$$
$$\sum_{m=1}^{\infty} (X_m + Y_m) = 1, \quad X_m \geq 0 \text{ and } Y_m \geq 0.$$

In particular, the extreme points of $WT_{\mathcal{H}}([\alpha_1], \gamma)$ *are* $\{h_m\}$ *and* $\{g_m\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_m(z)) \\ &= \sum_{m=1}^{\infty} (X_m + Y_m) z - \sum_{m=2}^{\infty} \frac{1-\gamma}{m(2m-1-\gamma)\sigma_m(\alpha_1)} X_m z^m \\ &\quad - \sum_{m=1}^{\infty} \frac{1-\gamma}{m(2m+1+\gamma)\sigma_m(\alpha_1)} Y_m \overline{z}^m \\ &= z - \sum_{m=2}^{\infty} A_m z^m - \sum_{m=1}^{\infty} B_m \overline{z}^m \end{aligned}$$

where $A_m = \frac{1-\gamma}{m(2m-1-\gamma))\sigma_m(\alpha_1)} X_m$, and $B_m = \frac{1-\gamma}{m(2m+1+\gamma)\sigma_m(\alpha_1)} Y_m$.
Therefore

$$\sum_{m=2}^{\infty} \frac{m(2m-1-\gamma)\sigma_m(\alpha_1)}{1-\gamma} A_m + \sum_{m=1}^{\infty} \frac{m(2m+1+\gamma)\sigma_m(\alpha_1)}{1-\gamma} B_m$$
$$= \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m = 1 - X_1 \leq 1,$$

and hence $f(z) \in clcoWT_{\mathcal{H}}([\alpha_1], \gamma)$.
Conversely, suppose that $f(z) \in clcoWT_{\mathcal{H}}([\alpha_1], \gamma)$. Setting

$$X_m = \frac{m(2m-1-\gamma)\sigma_m(\alpha_1)}{1-\gamma} A_m, \quad (m \geq 2)$$

and

$$Y_m = \frac{m(2m+1+\gamma)\sigma_m(\alpha_1)}{1-\gamma} B_m, \quad (m \geq 1)$$

where $\sum\limits_{m=1}^{\infty}(X_m+Y_m)=1$. Then

$$\begin{aligned} f(z) &= z-\sum_{m=2}^{\infty}a_m z^m-\sum_{m=1}^{\infty}\overline{b}_m\overline{z}^n, \quad (a_m,\ b_m\geq 0). \\ &= z-\sum_{m=2}^{\infty}\frac{1-\gamma}{m(2m-1-\gamma)\sigma_m(\alpha_1)}X_m z^m-\sum_{m=1}^{\infty}\frac{1-\gamma}{m(2m+1+\gamma)\sigma_m(\alpha_1)}Y_m\overline{z}^m \\ &= z-\sum_{m=2}^{\infty}(h_m(z)-z)X_m-\sum_{m=1}^{\infty}(g_m(z)-z)Y_m \\ &= \sum_{m=1}^{\infty}(X_m h_m(z)+Y_m g_m(z)) \end{aligned}$$

as required.

4.Inclusion Results

Now we show that $WT_{\mathcal{H}}([\alpha_1],\gamma)$ is closed under the convex combination of its members and is also closed under the convolution product.

Theorem 5. *The family $WT_{\mathcal{H}}([\alpha_1],\gamma)$ is closed under convex combinations.*

Proof. For $i=1,2,\ldots$, suppose that $f_i\in WT_{\mathcal{H}}([\alpha_1],\gamma)$ where

$$f_i(z)=z-\sum_{m=2}^{\infty}a_{i,m}z^m-\sum_{m=1}^{\infty}\overline{b}_{i,m}\overline{z}^m.$$

Then, by Theorem 2

$$\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}a_{i,m}+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}b_{i,m}\leq 1. \tag{18}$$

For $\sum\limits_{i=1}^{\infty}t_i$, $0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_i f_i(z)=z-\sum_{m=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i a_{i,m}\right)z^m-\sum_{m=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\overline{b}_{i,m}\right)\overline{z}^m.$$

Using the inequality (16), we obtain

$$\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_i a_{i,m}\right)+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_i b_{i,m}\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}a_{i,m}+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\sigma_m(\alpha_1)}{(1-\gamma)}b_{i,m}\right)$$
$$\leq\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum\limits_{i=1}^{\infty}t_i f_i\in WT_{\mathcal{H}}([\alpha_1],\gamma)$.

Theorem 6. *For* $0\leq\beta\leq\gamma<1$, *let* $f(z)\in WT_{\mathcal{H}}([\alpha_1],\gamma)$ *and* $F(z)\in WT_{\mathcal{H}}([\alpha_1],\delta)$. *Then* $f(z)*F(z)\in WT_{\mathcal{H}}([\alpha_1],\gamma)\subset WT_{\mathcal{H}}([\alpha_1],\delta)$.

Proof. Let $f(z)=z-\sum\limits_{m=2}^{\infty}a_m z^m-\sum\limits_{m=1}^{\infty}\bar{b}_m\bar{z}^m\in WT_{\mathcal{H}}([\alpha_1],\gamma)$ and $F(z)=z-\sum\limits_{m=2}^{\infty}A_m z^m-\sum\limits_{m=1}^{\infty}\overline{B}_m\overline{z}^m\in WT_{\mathcal{H}}([\alpha_1],\delta)$. Then $f(z)*F(z)$ is $f(z)*F(z)=z-\sum\limits_{m=2}^{\infty}a_m A_m z^m-\sum\limits_{m=1}^{\infty}\bar{b}_m\overline{B}_m\overline{z}^m$.

For $f(z)*F(z)\in WT_{\mathcal{H}}([\alpha_1],\delta)$ we note that $|A_m|\leq 1$ and $|B_m|\leq 1$. Now by Theorem 2, we have

$$\sum_{m=2}^{\infty}\frac{m(2m-1-\delta)\sigma_m(\alpha_1)}{1-\delta}|a_m|\ \ |A_m|+\sum_{m=1}^{\infty}\frac{m(2m+1+\delta)\sigma_m(\alpha_1)}{1-\delta}|b_m|\ \ |B_m|$$
$$\leq\ \sum_{m=2}^{\infty}\frac{m(2m-1-\delta)\sigma_m(\alpha_1)}{1-\delta}|a_m|+\sum_{m=1}^{\infty}\frac{m(2m+1+\delta))\sigma_m(\alpha_1)}{1-\delta}|b_m|$$

and since $0\leq\delta\leq\gamma<1$

$$\leq\ \sum_{m=2}^{\infty}\frac{m(2m-1-\gamma))\sigma_m(\alpha_1)}{1-\gamma}|a_m|+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma))\sigma_m(\alpha_1)}{1-\gamma}|b_m|\leq 1,$$

by Theorem 2 $f(z)\in WT_{\mathcal{H}}([\alpha_1],\gamma)$. Therefore $f(z)*F(z)\in WT_{\mathcal{H}}([\alpha_1],\gamma)\subset WT_{\mathcal{H}}([\alpha_1],\delta)$.

Lastly, we examine the closure properties of the class $WT_{\mathcal{H}}([\alpha_1],\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$L_c(f)$ which is defined by

$$L_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\quad(c>-1).$$

Theorem 7. *Let $f(z) \in WT_{\mathcal{H}}([\alpha_1], \gamma)$, then $L_c(f(z)) \in WT_{\mathcal{H}}([\alpha_1], \gamma)$*

Proof. From the representation of $L_c(f(z))$, it follows that

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt.$$

$$= \frac{c+1}{z^c} \left(\int_0^z t^{c-1} \left(t - \sum_{m=2}^{\infty} a_m t^m \right) dt - \overline{\int_0^z t^{c-1} \left(\sum_{m=1}^{\infty} b_m t^m \right) dt} \right)$$
$$= z - \sum_{m=2}^{\infty} A_m z^m - \sum_{n=1}^{\infty} B_m z^m$$

where $A_m = \frac{c+1}{c+m}\ a_m; B_m = \frac{c+1}{c+m}\ b_m$. Therefore,

$$\sum_{m=1}^{\infty} m \left(\frac{2m-1-\gamma}{1-\gamma} (\frac{c+1}{c+m} |a_m|) + \frac{2m+1+\gamma}{1-\gamma} (\frac{c+1}{c+m} |b_m|) \right) \sigma_m(\alpha_1)$$
$$\leq \sum_{m=1}^{\infty} m \left(\frac{2m-1-\gamma}{1-\gamma} |a_m| + \frac{2m+1+\gamma}{1-\gamma} |b_m| \right) \sigma_m(\alpha_1)$$
$$\leq 2(1-\gamma).$$

Since $f(z) \in WT_{\mathcal{H}}([\alpha_1], \gamma)$, therefore by Theorem 2, $L_c(f(z)) \in WT_{\mathcal{H}}([\alpha_1], \gamma)$.

Concluding Remarks. We observe that if we specialize the various parameters of the class $WT_{\mathcal{H}}([\alpha_1], \gamma)$ suitably, we would arrive at the analogous results for the classes $HG_{\mathcal{H}}([\alpha_1], \gamma)$, $LG_{\mathcal{H}}([\alpha_1], \gamma)$ $RG_{\mathcal{H}}([\alpha_1], \gamma)$ and $FG_{\mathcal{H}}([\alpha_1], \gamma)$ (defined above in Section 1). These obvious consequences of our results being straightforward, further details are hence omitted here.

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