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A CLASS OF INTEGRAL OPERATORS PRESERVING DOUBLE SUBORDINATIONS

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Abstract. In the present paper, we investigate some subordination-preserving and superordination-preserving properties of a class of integral operators which are defined on the space of meromorphic functions. Several sandwich-type results involving this class of integral operators are also derived.

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1. Introduction and preliminaries

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : \ z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} := \mathbb{U} \setminus \{0\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \{ \mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \ \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial \mathbb{U} : \ \lim_{z \to \varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U}\backslash E(f)$. The subclass of Q for which $f(0) = a$ ($a \in \mathbb{C}$) is denoted by $Q(a)$.

Let $f,\ g \in \Sigma$, where f is given by (1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z) := \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k =: (g*f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

By setting

$$f_n(z) := \frac{1}{z(1-z)^{n+1}} \qquad (n > 1) \tag{2}$$

and let $f^*_{n,\mu}(z)$ be so defined that

$$f_n(z) * f^*_{n,\mu}(z) := \frac{1}{z(1-z)^{\mu}} \qquad (\mu > 0). \tag{3}$$

In a recent paper, Yuan *et al.* [6] defined a class of integral operators as follows:

$$\mathcal{I}_{n,\mu}(z) := f^*_{n,\mu}(z) * f(z) \qquad (f \in \Sigma), \tag{4}$$

where (and throughout this paper unless otherwise mentioned) the parameters n and μ are constrained as $n > -1$ and $\mu > 0$.

We can easily find from (2), (3) and (4) that

$$\mathcal{I}_{n,\mu}(z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(n+1)_{k+1}} a_k z^k \qquad (z \in \mathbb{U}), \tag{5}$$

where $(\kappa)m$ is the Pochhammer symbol defined by

$$(\kappa)_0 = 1 \quad \text{and} \quad (\kappa)_m = \upsilon(\upsilon+1)\cdots(\upsilon+m-1) \qquad (k \in \mathbb{N}).$$

It is readily verified from (5) that

$$z\left(\mathcal{I}_{n+1,\mu}f\right)'(z) = (n+1)\mathcal{I}_{n,\mu}f(z) - (n+2))\mathcal{I}_{n+1,\mu}f(z), \tag{6}$$

and

$$z\left(\mathcal{I}_{n,\mu}f\right)'(z) = \mu\mathcal{I}_{n,\mu+1}f(z) - (\mu+1)\mathcal{I}_{n,\mu}f(z). \tag{7}$$

In order to prove our main results, we need the following lemmas.

Lemma 1. (see [1]) *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *for all real* s *and for all*

$$t \leqq -\frac{n\left(1+s^2\right)}{2} \qquad (n \in \mathbb{N})$$

satisfies the condition $\Re(H(is,t)) \leqq 0$. *If the function*

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ *and*

$$\Re\left(H(p(z), zp'(z))\right) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 2. (see [2]) *Let* $\kappa,\ \gamma \in \mathbb{C}$ *with* $\kappa \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If*

$$\Re(\kappa h(z) + \gamma) > 0 \qquad (z \in \mathbb{U}),$$

then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \qquad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ *and satisfies the inequality given by*

$$\Re(\kappa q(z) + \gamma) > 0 \qquad (z \in \mathbb{U}).$$

Lemma 3. (see [3]) *Let* $p \in Q(a)$ *and*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (q \neq a;\ n \in \mathbb{N})$$

be analytic in $\mathbb{U}$*. If* q *is not subordinate to* p*, then there exists two points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad and \quad \xi_0 \in \partial\mathbb{U} \backslash E(f)$$

such that

$$q(\mathbb{U}_{r_0}) \subset p\,(\mathbb{U}), \quad q(z_0) = p\,(\xi_0) \quad and \quad z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geqq n).$$

A function $P(z,t)$ $(z \in \mathbb{U};\ t \geqq 0)$ is said to be a subordination chain if $P(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqq 0$, $P(z,0)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $P(z,t_1) \prec P(z,t_2)$ for all $0 \leqq t_1 \leqq t_2$.

Lemma 4. (see [4]) *The function* $P(z,t):\ \mathbb{U} \times [0,\infty) \to \mathbb{C}$ *of the form*

$$P(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \qquad (a_1(t) \neq 0;\ t \geqq 0)\,,$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\Re\left(\frac{z\,\partial P/\partial z}{\partial P/\partial t}\right) > 0 \qquad (z \in \mathbb{U};\ t \geqq 0).$$

Lemma 5. (see [5]) *Let* $q \in \mathcal{H}[a,1]$ *and* $\phi : \mathbb{C}^2 \to \mathbb{C}$*. Also set*

$$\phi\left(q(z), zq'(z)\right) \equiv h(z) \qquad (z \in \mathbb{U}).$$

If $P(z,t) := \phi\left(q(z), tzq'(z)\right)$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap Q(a)$*. Then*

$$h(z) \prec \phi\left(p(z), zp'(z)\right) \qquad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if $\phi\left(q(z), zq'(z)\right) = h(z)$ *has a univalent solution* $q \in Q(a)$*, then* q *is the best subordination.*

The main purpose of the present paper is to investigate some subordination-preserving and superordination-preserving properties associated with the operator $I_{n,\mu}$. Several sandwich-type results involving this operator are also derived.

2. Main Results

We begin by stating the following subordination result involving the operator $\mathcal{I}_{n,\mu}$.

Theorem 1. *Let $f,\ g \in \Sigma$ and $n > -1$. If*

$$\Re\left(1+\frac{z\psi''(z)}{\psi'(z)}\right) > -\rho \qquad \left(z \in \mathbb{U};\ \psi(z) := z\mathcal{I}_{n,\mu}g(z)\right), \tag{8}$$

where

$$\rho := \frac{1+(n+1)^2 - \left|1-(n+1)^2\right|}{4(n+1)}, \tag{9}$$

then the following subordination relationship

$$z\mathcal{I}_{n,\mu}f(z) \prec z\mathcal{I}_{n,\mu}g(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n+1,\mu}f(z) \prec z\mathcal{I}_{n+1,\mu}g(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the function $\mathcal{I}_{n+1,\mu}g(z)$ is the best dominant.

Proof. Let us define the functions $\mathcal{F}$ and $\mathcal{G}$ by

$$\mathcal{F}(z) := z\mathcal{I}_{n+1,\mu}f(z), \quad \mathcal{G}(z) := z\mathcal{I}_{n+1,\mu}g(z). \tag{10}$$

We here assume, without loss of generality, that $\mathcal{G}$ is analytic and univalent on $\overline{\mathbb{U}}$ and

$$\mathcal{G}'(\zeta) \neq 0 \qquad (|\zeta| = 1).$$

If not, then we replace $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F}(\rho z)$ and $\mathcal{G}(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, and we can use them in the proof of our result. Therefore, our results would follow by letting $\rho \to 1$.

We first show that if the function $\mathcal{Q}$ be defined by

$$\mathcal{Q}(z) := 1 + \frac{z\mathcal{G}''(z)}{\mathcal{G}'(z)} \qquad (z \in \mathbb{U}), \tag{11}$$

then

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

By virtue of (6) and the definitions of $\mathcal{G}$ and ψ, we know that

$$\psi(z) = \mathcal{G}(z) + \frac{1}{n+1}\, z\mathcal{G}'(z). \tag{12}$$

Differentiating both sides of (12) with respect to z, we get

$$\psi'(z) = \left(1 + \frac{1}{n+1}\right)\mathcal{G}'(z) + \frac{1}{n+1} z\mathcal{G}''(z). \tag{13}$$

After some simple calculations, in conjunction with (11) and (13), we easily get the relationship

$$1 + \frac{z\psi''(z)}{\psi'(z)} = \mathcal{Q}(z) + \frac{z\mathcal{Q}'(z)}{\mathcal{Q}(z)+n+1} := \mathbb{h}(z) \qquad (z \in \mathbb{U}). \tag{14}$$

We also deduce from (8) and (14) that

$$\Re\left(\mathbb{h}(z) + n + 1\right) > 0 \qquad (z \in \mathbb{U}). \tag{15}$$

Furthermore, by Lemma 2, we conclude that the differential equation (14) has a solution $\mathcal{Q} \in \mathcal{H}(\mathbb{U})$ with

$$\mathbb{h}(0) = \mathcal{Q}(0) = 1.$$

Let us put

$$H(u,v) := u + \frac{v}{u+n+1} + \rho, \tag{16}$$

where ρ is given by (9). From (14), (15) and (16), we obtain

$$\Re\left(H(\mathcal{Q}(z), z\mathcal{Q}'(z))\right) > 0 \qquad (z \in \mathbb{U}).$$

Now we proceed to show that

$$\Re(H(is,t)) \leqq 0 \qquad \left(s \in \mathbb{R};\ t \leqq -\frac{1+s^2}{2}\right), \tag{17}$$

Indeed, from (16), we have

$$\Re(H(is,t)) = \Re\left(is + \frac{t}{is+n+1} + \rho\right) = \frac{(n+1)t}{|n+1+is|^2} + \rho \leqq -\frac{\Psi(n,s)}{2\,|n+1+is|^2},$$

where

$$\Psi(n,s) := (n+1-2\rho)s^2 - 4\rho(n+1)s - 2\rho(n+1)^2 + n + 1. \tag{18}$$

For ρ given by (9), the coefficient of s^2 in the quadratic expression $\Psi(n,s)$ given by (18) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\Psi(n,s)$ by s in (18) is a perfect square, which implies that (17) holds. Thus, by Lemma 1, we conclude that

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

By the definition of $\mathcal{G}(z)$, we know that $\mathcal{G}(z)$ is convex.

To prove $\mathcal{F} \prec \mathcal{G}$, we let the function $P(z,t)$ be defined by

$$P(z,t) := \mathcal{G}(z) + \left(\frac{1+t}{n+1}\right) z\mathcal{G}'(z) \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty), \tag{19}$$

since $\mathcal{G}$ is convex and $\lambda > 0$, we have

$$\frac{\partial P(z,t)}{\partial z}|_{z=0} = \mathcal{G}'(0)\left(1 + \frac{1+t}{n+1}\right) \neq 0 \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty),$$

and

$$\Re\left(\frac{z\,\partial P(z,t)/\partial z}{\partial P(z,t)/\partial t}\right) = (1+t)\Re\left(\mathcal{Q}(z)\right) + n + 1 > 0 \qquad (z \in \mathbb{U}).$$

Therefore, by Lemma 4, we obtain that $P(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\psi(z) = \mathcal{G}(z) + \frac{1}{n+1} z\mathcal{G}'(z) = P(z,0),$$

and

$$P(z,0) \prec P(z,t) \qquad (z \in \mathbb{U};\ 0 \leqq t < \infty),$$

which implies that

$$P(\zeta,t) \notin P(\mathbb{U},0) = \psi(\mathbb{U}) \qquad (\zeta \in \partial\mathbb{U};\ 0 \leqq t < \infty). \tag{20}$$

If $\mathcal{F}$ is not subordinate to $\mathcal{G}$, by Lemma 3, we know that there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$\mathcal{F}(z_0) = \mathcal{G}(\zeta_0) \quad \text{and} \quad z_0\mathcal{F}'(z_0) = (1+t)\zeta_0\mathcal{G}'(\zeta_0) \quad (0 \leqq t < \infty). \tag{21}$$

Hence, by virtue of (7) and 21, we have

$$P(\zeta_0,t) = \mathcal{G}(\zeta_0) + \frac{1+t}{n+1}\zeta_0\mathcal{G}'(\zeta_0) = \mathcal{F}(z_0) + \frac{1}{n+1} z_0\mathcal{F}'(z_0) = z_0\mathcal{I}_{n,\mu}f(z_0) \in \psi(\mathbb{U}).$$

But this contradicts to (20). Thus, we deduce that $\mathcal{F} \prec \mathcal{G}$. Considering $\mathcal{F} = \mathcal{G}$, we see that the function $\mathcal{G}$ is the best dominant. The proof of Theorem 1 is evidently completed. □

By similarly applying the method of proof of Theorem 1 and using (7), we easily get the following result.

Corollary 1. *Let $f,\ g \in \Sigma$ and $\mu > 0$. If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad (z \in \mathbb{U};\ \varphi(z) := z\mathcal{I}_{n,\mu+1}g(z)),$$

where

$$\eta := \frac{1+\mu^2-\left|1-\mu^2\right|}{4\mu}, \tag{22}$$

then the following subordination relationship

$$z\mathcal{I}_{n,\mu+1}f(z) \prec z\mathcal{I}_{n,\mu+1}g(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n,\mu}f(z) \prec z\mathcal{I}_{n,\mu}g(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the function $z\mathcal{I}_{n,\mu}g(z)$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result.

Theorem 2. *Let $f,\ g \in \Sigma$ and $n > -1$. If*

$$\Re\left(1+\frac{z\psi''(z)}{\psi'(z)}\right) > -\rho \qquad (z \in \mathbb{U};\ \psi(z) := z\mathcal{I}_{n,\mu}g(z)),$$

where ρ is given by (9), also let the function $z\mathcal{I}_{n,\mu}g(z)$ is univalent in $\mathbb{U}$ and $z\mathcal{I}_{n+1,\mu}g(z) \in Q$, then the following subordination relationship

$$z\mathcal{I}_{n,\mu}g(z) \prec z\mathcal{I}_{n,\mu}f(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n+1,\mu}g(z) \prec z\mathcal{I}_{n+1,\mu}f(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the function $z\mathcal{I}_{n+1,\mu}g(z)$ is the best subdominant.

Proof. Suppose that the functions $\mathcal{F}$ and $\mathcal{G}$ are defined by (10), $\mathcal{Q}$ is defined by (11). By applying the similar method as in the proof of Theorem 1, we get

$$\Re(\mathcal{Q}(z)) > 0 \qquad (z \in \mathbb{U}).$$

Next, to arrive at our desired result, we show that $\mathcal{G} \prec \mathcal{F}$. For this, we suppose that the function $P(z,t)$ be defined by (19). Since $n > -1$ and $\mathcal{G}$ is convex, by applying the similar method as in Theorem 1, we deduce that $P(z,t)$ is subordination chain. Therefore, by Lemma 5, we conclude that $\mathcal{G} \prec \mathcal{F}$. Furthermore, since the differential equation

$$\psi(z) = \mathcal{G}(z) + \frac{1}{n+1}z\mathcal{G}'(z) := \phi\left(\mathcal{G}(z), z\mathcal{G}'(z)\right)$$

has a univalent solution $\mathcal{G}$, it is the best subordination. We thus complete the proof of Theorem 2. □

By similarly applying the method of proof of Theorem 2 and using (7), we easily get the following result.

Corollary 2. *Let* $f,\ g \in \Sigma$ *and* $\mu > 0$. *If*

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\eta \qquad (z \in \mathbb{U};\ \varphi(z) := z\mathcal{I}_{n,\mu+1}g(z)),$$

where η *is given by* (22), *also let the function* $z\mathcal{I}_{n,\mu+1}g(z)$ *is univalent in* $\mathbb{U}$ *and* $z\mathcal{I}_{n,\mu}g(z) \in Q$, *then the following subordination relationship*

$$z\mathcal{I}_{n,\mu+1}g(z) \prec z\mathcal{I}_{n,\mu+1}f(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n,\mu}g(z) \prec z\mathcal{I}_{n,\mu}f(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the function $z\mathcal{I}_{n,\mu}g(z)$ *is the best subdominant.*

Combining the above mentioned subordination and superordination results, we get the following "sandwich-type result".

Corollary 3. *Let* $f,\ g_k \in \Sigma$ $(k = 1, 2)$ *and* $n > -1$. *If*

$$\Re\left(1+\frac{z\psi_k''(z)}{\psi_k'(z)}\right) > -\rho \qquad (z \in \mathbb{U};\ \psi_k(z) := z\mathcal{I}_{n,\mu}g_k(z) \ \ (k = 1, 2)),$$

where ρ *is given by* (9), *also let the function* $z\mathcal{I}_{n,\mu}f(z)$ *is univalent in* $\mathbb{U}$ *and* $z\mathcal{I}_{n+1,\mu}f(z) \in Q$, *then the following subordination relationship*

$$z\mathcal{I}_{n,\mu}g_1(z) \prec z\mathcal{I}_{n,\mu}f(z) \prec z\mathcal{I}_{n,\mu}g_2(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n+1,\mu}g_1(z) \prec z\mathcal{I}_{n+1,\mu}f(z) \prec z\mathcal{I}_{n+1,\mu}g_2(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the functions $z\mathcal{I}_{n+1,\mu}g_1(z)$ *and* $z\mathcal{I}_{n+1,\mu}g_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Corollary 4. *Let* $f,\ g_k \in \Sigma$ $(k = 1, 2)$ *and* $\mu > 0$. *If*

$$\Re\left(1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) > -\eta \qquad (z \in \mathbb{U};\ \varphi_k(z) := z\mathcal{I}_{n,\mu+1}g_k(z) \ \ (k = 1, 2)),$$

where η *is given by* (22), *also let the function* $z\mathcal{I}_{n,\mu+1}f(z)$ *is univalent in* $\mathbb{U}$ *and* $z\mathcal{I}_{n,\mu}f(z) \in Q$, *then the following subordination relationship*

$$z\mathcal{I}_{n,\mu+1}g_1(z) \prec z\mathcal{I}_{n,\mu+1}f(z) \prec z\mathcal{I}_{n,\mu+1}g_2(z) \qquad (z \in \mathbb{U})$$

implies that

$$z\mathcal{I}_{n,\mu}g_1(z) \prec z\mathcal{I}_{n,\mu}f(z) \prec z\mathcal{I}_{n,\mu}g_2(z) \qquad (z \in \mathbb{U}).$$

Furthermore, the functions $z\mathcal{I}_{n,\mu}g_1(z)$ *and* $z\mathcal{I}_{n,\mu}g_2(z)$ *are, respectively, the best subordinant and the best dominant.*

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