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# SUFFICIENT CONDITIONS FOR UNIVALENCE OF INTEGRAL OPERATOR DEFINED BY HADAMARD PRODUCT

B.A. Frasin

Abstract. In this paper, we obtain new sufficient conditions for the univalence of general integral operator defined by

$$I_\beta(f_1,...,f_n;g_1,...,g_n)(z)=\left\{\int_0^z \beta t^{\beta-1}\left(\frac{(f_1*g_1)(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{(f_n*g_n)(t)}{t}\right)^{\alpha_n}dt\right\}^{\frac{1}{\beta}}.$$

Several corollaries and consequences of the main results are also considered.

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## 1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disc $\mathcal{U}=\{z:|z|<1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. For two functions $f(z)\in\mathcal{A}$ and $g(z)$ given by

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n \tag{1}$$

their Hadamard product (or convolution) is defined by

$$(f*g)(z):=z+\sum_{n=2}^{\infty}a_nb_nz^n. \tag{2}$$

For a function $g \in \mathcal{A}$ defined by (1), where $b_n \geq 0 \ (n \geq 2),$ we define the family $\mathcal{S}(g,\gamma)$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right| < 1 - \gamma \qquad (z \in \mathcal{U};\ 0 \leq \gamma < 1), \tag{3}$$

provided that $(f*g)(z) \neq 0.$

Also, for a function $g \in \mathcal{A}$ defined by (1), where $b_n \geq 0 \ (n \geq 2),$ we define the family $\mathcal{B}(g,\mu)$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \frac{z^2(f*g)'(z)}{[(f*g)(z)]^2} - 1 \right| < 1 - \mu \qquad (z \in \mathcal{U};\ 0 \leq \mu < 1), \tag{4}$$

provided that $(f*g)(z) \neq 0.$

Note that $\mathcal{B}(\frac{z}{1-z},\mu) = \mathcal{B}(\mu)$, where the class $\mathcal{B}(\mu)$ of analytic and univalent functions was introduced and studied by Frasin and Darus [11](see also [10]).

Using the Hadamard product defined by (2), we introduce the following general integral operator:

**Definition 1.** *Given* $f_i\,, g_i \in \mathcal{A},\ \alpha_i \in \mathbb{C}$ *for all* $i = 1, \ldots, n,\ n \in \mathbb{N},\ \beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) > 0.$ *We let* $I_\beta(f_1,...,f_n; g_1,...,g_n) : \mathcal{A}^n \to \mathcal{A}$ *be the integral operator defined by*

$$I_\beta(f_1,...,f_n;g_1,...,g_n)(z) = \left\{ \int_0^z \beta t^{\beta-1} \left( \frac{(f_1*g_1)(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{(f_n*g_n)(t)}{t} \right)^{\alpha_n} dt \right\}^{\frac{1}{\beta}} \tag{5}$$

*where* $(f*g)(z)/z \neq 0,\ \ z \in \mathcal{U}.$

Here and throughout in the sequel every many-valued function is taken with the principal branch.

**Remark 1.** Note that the integral operator $I_\beta(f_1,...,f_n;g_1,...,g_n)(z)$ generalizes many operators introduced and studied by several authors, for example:

(1) For $\beta = 1,$ we obtain the integral operator

$$I(f_1,...,f_n;g_1,...,g_n)(z) = \int_0^z \left( \frac{(f_1*g_1)(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{(f_n*g_n)(t)}{t} \right)^{\alpha_n} dt \tag{6}$$

introduced and studied by Frasin [9].

(2) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}(1+(n-1))\lambda\right)^m z^n$, we obtain the following integral operator introduced and studied by Bulut [7]

$$I_{\beta}^{m,\eta}(f_1,...,f_n)(z) = \left\{\int_0^z \beta t^{\beta-1}\left(\frac{D_{\lambda}^{m,\eta}f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{D_{\lambda}^{m,\eta}f_n(t)}{t}\right)^{\alpha_n} dt\right\}^{\frac{1}{\beta}} \tag{7}$$

where $D_{\lambda}^{m,\eta}f(\ z) = z + \sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}(1+(n-1))\lambda\right)^m a_n z^n$ , $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ is the generalized Al-Oboudi operator [2].

(3) For $g_1 = \cdots = g_n = \frac{z}{1-z}$, we obtain the integral operator

$$I_{\beta}(f_1,...,f_n)(z) = \left\{\int_0^z \beta t^{\beta-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt\right\}^{\frac{1}{\beta}} \tag{8}$$

introduced and studied by Breaz and Breaz [3].

(4) For $g_1 = \cdots = g_n = \frac{z}{1-z}$ and $\beta = 1,$ we obtain the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{9}$$

introduced and studied by Breaz and Breaz [3].

(5) For $g_1 = \cdots = g_n = \frac{z}{(1-z)^2}$ and $\beta = 1$,we obtain the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1}\cdots\left(f_n'(t)\right)^{\alpha_n} dt \tag{10}$$

introduced and studied by Breaz *et al.* [5].

(6) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} C_{k+n-1}^{k} z^n$ and $\beta = 1$,we obtain the following integral operator introduced in [12]

$$I(f_1,...,f_n)(z) = \int_0^z \left(\frac{R^k f_1(t)}{t}\right)^{\alpha_1}\cdots\left(\frac{R^k f_n(t)}{t}\right)^{\alpha_n} dt \tag{11}$$

where $R^k f(z) = z + \sum_{n=2}^{\infty} C_{k+n-1}^{k} a_n z^n$, $k \in \mathbb{N}_0$ is Ruscheweyh differential operator [18].

(7) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} n^k z^n$ and $\beta = 1$,we obtain the following integral operator introduced and studied by Breaz *et al.* [4]

$$D^k F(z) = \int_0^z \left( \frac{D^k f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{D^k f_n(t)}{t} \right)^{\alpha_n} dt \tag{12}$$

where $D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ k \in \mathbb{N}_0$ is Sãlãgean differential operator [19].

(8) $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k z^n$ and $\beta = 1$,we obtain the following integral operator introduced and studied by Bulut [6]

$$I_n(f_1, ..., f_n)(z) = \int_0^z \left( \frac{D_\lambda^k f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{D_\lambda^k f_n(t)}{t} \right)^{\alpha_n} dt \tag{13}$$

where $D_\lambda^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k a_n z^n, 0 \leq \lambda \leq 1$, is Al-Oboudi differential operator [2].

(9) For $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$ and $\beta = 1$,we obtain the integral operator introduced and studied by Selvaraj and Karthikeyan [20]

$$F_\alpha(a, c; z) = \int_0^z \left( \frac{L(a,c) f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{L(a,c) f_n(t)}{t} \right)^{\alpha_n} dt \tag{14}$$

where $L(a, c) f(z) := z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$ is the Carlson-Shaffer linear operator [8].

(10) For $g_1 = \frac{z}{1-z}$ and $\alpha_1 = \beta = 1,$ we obtain Alexander integral operator introduced in [1]

$$I(z) = \int_0^z \frac{f_1(t)}{t} dt \tag{15}$$

(11) For $g_1 = \frac{z}{1-z}$ , $\alpha_1 = \alpha,$ and $\beta = 1$,we obtain the integral operator

$$F_\alpha(z) = \int_0^z \left( \frac{f(t)}{t} \right)^{\alpha} dt \tag{16}$$

studied in [13].

In order to derive our main results, we have to recall here the following univalence criteria.

**Lemma 1.** ([15]) Let $\alpha$ be a complex number with $\mathrm{Re}(\alpha) > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

*for all* $z \in \mathcal{U}$, *then the integral operator*

$$F_\alpha(z) = \left\{\alpha \int_0^z t^{\alpha-1} f'(t)dt\right\}^{\frac{1}{\alpha}}$$

*is in the class* $\mathcal{S}$.

**Lemma 2.** ([16]) *Let* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

*for all* $z \in \mathcal{U}$, *then, for any complex number* $\beta$ *with* $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$, *the integral operator*

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)dt\right\}^{\frac{1}{\beta}}$$

*is in the class* $\mathcal{S}$.

**Lemma 3.** ([17]) *Let* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) > 0$, $c \in \mathbb{C}$ *with* $|c| \leq 1$, $c \neq -1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left|c\,|z|^{2\beta} + (1-|z|^{2\beta})\frac{zf''(z)}{\beta f'(z)}\right| \leq 1,$$

*for all* $z \in \mathcal{U}$, *then the integral operator*

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)dt\right\}^{\frac{1}{\beta}}$$

*is in the class* $\mathcal{S}$.

Also, we need the following general Schwarz Lemma

**Lemma 4.**([14]) *Let the function $f$ be regular in the disk $\mathcal{U}_R = \{z : |z| < R\}$, with $|f(z)| < M$ for fixed $M$. If $f(z)$ has one zero with multiplicity order bigger than $m$ for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \qquad (z \in \mathcal{U}_R).$$

*The equality can hold only if*

$$f(z) = e^{i\theta} (M/R^m) z^m$$

*where $\theta$ is constant.*

In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $I_\beta(f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (2). Several corollaries and consequences of the main results are also considered.

## 2. Univalence conditions for $I_\beta(f_1, ..., f_n; g_1, ..., g_n)$

We first prove the following theorem.

**Theorem 1.** *Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and $\beta \in \mathbb{C}$ with*

$$\operatorname{Re}(\beta) \geq \sum_{i=1}^{n} |\alpha_i| (1 - \gamma_i) > 0. \tag{17}$$

*If $f_i \in \mathcal{S}(g_i, \gamma_i), 0 \leq \gamma_i < 1$ for all $i = 1, \ldots, n$, then the integral operator $I_\beta(f_1, ..., f_n; g_1, ..., g_n)$ defined by (5) is analytic and univalent in $\mathcal{U}$.*

*Proof.* Define

$$h(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{(f_i * g_i)(t)}{t} \right)^{\alpha_i} dt,$$

thus we have

$$h'(z) = \prod_{i=1}^{n} \left( \frac{(f_i * g_i)(z)}{z} \right)^{\alpha_i}. \tag{18}$$

Differentiating both sides of (18) with respect to $z$ logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right)$$

thus we have

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} |\alpha_i| \left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right|. \tag{19}$$

Since $f_i \in \mathcal{S}(g_i, \gamma_i)$ for all $i = 1, \ldots, n$, from (3), (17) and (19), we obtain

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}|\alpha_i|\left|\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}-1\right| \\ &\leq \frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}|\alpha_i|(1-\gamma_i) \\ &\leq 1.\end{aligned}$$

Applying Lemma 1 for the function $h(z)$, we prove that $I_\beta(f_1, ..., f_n; g_1, ..., g_n) \in \mathcal{S}$.

Letting $g_1 = \cdots = g_n = z + \sum\limits_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}(1+(n-1))\lambda\right)^m z^n$ and $\gamma_i = 0,$ for all $i = 1, \ldots, n,$ in Theorem 1, we have:

**Corollary 1.** ([7])*Let $\alpha_i \in \mathbb{C}$ for all $i = 1, \ldots, n$ and $\beta \in \mathbb{C}$ with*

$$\mathrm{Re}(\beta) \geq \sum_{i=1}^{n}|\alpha_i| > 0.$$

*If*

$$\left|\frac{z(D_\lambda^{m,\eta}f_i(z))'}{D_\lambda^{m,\eta}f_i(z)}-1\right| < 1 \qquad (z \in \mathcal{U}, m \in \mathbb{N}_0)$$

*then the integral operator $I_\beta^{m,\eta}(f_1, ..., f_n)(z)$ defined by (7) is analytic and univalent in $\mathcal{U}$.*

Making use of Lemma 2, we prove the following theorem.

**Theorem 2.** *Let $\alpha_i \in \mathbb{C}$ , $M_i \geq 1$ for all $i = 1, \ldots, n$ and $\beta \in \mathbb{C}$ with*

$$\mathrm{Re}(\beta) \geq \sum_{i=1}^{n}|\alpha_i|((2-\mu_i)M_i+1) > 0. \tag{20}$$

*If $f_i \in \mathcal{B}(g_i, \mu_i), 0 \leq \mu_i < 1$ for all $i = 1, \ldots, n$ , and*

$$|(f_i * g_i)(z)| \leq M_i \ \ (z \in \mathcal{U}, i = 1, \ldots, n),$$

*then for any complex number $\beta$ with $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$, the integral operator $I_\beta(f_1, ..., f_n; g_1, ..., g_n)$ defined by (5) is analytic and univalent in $\mathcal{U}$.*

*Proof.* From the proof of Theorem 1, we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n}|\alpha_i|\left|\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}-1\right|$$

which readily shows that

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}\right|+1\right)$$
$$\leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}\right|\left|\frac{(f_i*g_i)(z)}{z}\right|+1\right).$$

Since $|(f_i*g_i)(z)|\leq M_i$ $(z\in\mathcal{U},\ i=1,\dots,n),$ and $f_i\in\mathcal{B}(g_i,\mu_i)$ for all $i=1,\dots,n,$ we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}-1\right|M_i+M_i+1\right)$$
$$\leq \frac{1}{\mathrm{Re}(\alpha)}\sum_{i=1}^{n}|\alpha_i|\left((2-\mu_i)M_i+1\right)\quad (z\in\mathcal{U}),$$

which, in the light of the hypothesis (20), yields

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right|\leq 1\quad (z\in\mathcal{U}).$$

Applying Lemma 2 for the function $h(z)$, we prove that $I_\beta(f_1,...,f_n;g_1,...,g_n)\in\mathcal{S}$.

Letting $g_1=\cdots=g_n=z+\sum\limits_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}(1+(n-1))\lambda\right)^m z^n$ and $\mu_i=0,$ for all $i=1,\dots,n,$ in Theorem 2, we have:

**Corollary 2.** ([7])*Let* $\alpha_i\in\mathbb{C}$ , $M_i\geq 1$ *for all* $i=1,\dots,n$ *and* $\beta\in\mathbb{C}$ *with*

$$\mathrm{Re}(\beta)\geq\sum_{i=1}^{n}|\alpha_i|\,(2M_i+1)>0.$$

*If*

$$\left|\frac{z^2(D_\lambda^{m,\eta}f_i(z))'}{(D_\lambda^{m,\eta}f_i(z))^2}-1\right|<1\qquad (z\in U, m\in N_0)$$

*and*

$$\left|D_\lambda^{m,\eta}f_i(z)\right|\leq M_i\ \ (z\in\mathcal{U}, i=1,\dots,n),$$

*then for any complex number* $\beta$ *with* $\mathrm{Re}(\beta)\geq\mathrm{Re}(\alpha)$, *the integral operator* $I_\beta^{m,\eta}(f_1,...,f_n)(z)$ *defined by (7) is analytic and univalent in* $\mathcal{U}$.

Next, we prove

**Theorem 3.** *Let* $\alpha_i \in \mathbb{C}$ *for all* $i = 1, \ldots, n$ *and* $\beta \in \mathbb{C}$ *with*

$$\mathrm{Re}(\beta) \geq \sum_{i=1}^{n} |\alpha_i| (1 - \gamma_i) > 0$$

*and let* $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i| (1 - \gamma_i).$$

*If* $f_i \in \mathcal{S}(g_i, \gamma_i), 0 \leq \gamma_i < 1$ *for all* $i = 1, \ldots, n$*, then the integral operator* $I_\beta(f_1, ..., f_n; g_1, ..., g_n)$ *defined by (5) is analytic and univalent in* $\mathcal{U}$*.*

*Proof.* From the proof of Theorem 1, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left[ \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right]. \tag{21}$$

Thus, we have

$$\begin{aligned} \left| c\,|z|^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \right| &= \left| c\,|z|^{2\beta} + (\frac{1 - |z|^{2\beta}}{\beta}) \sum_{i=1}^{n} \alpha_i \left[ \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right] \right| \\ &\leq |c| + \left| \frac{1 - |z|^{2\beta}}{\beta} \right| \sum_{i=1}^{n} |\alpha_i| \left| \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right| \\ &\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^{n} |\alpha_i| (1 - \gamma_i) \\ &\leq |c| + \frac{1}{\mathrm{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i| (1 - \gamma_i) \\ &\leq 1. \end{aligned}$$

Finally, by applying Lemma 3, we conclude that $I_\beta(f_1, ..., f_n; g_1, ..., g_n) \in \mathcal{S}$.

Letting $g_1 = \cdots = g_n = z + \sum_{n=2}^{\infty} \left( \frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)} (1 + (n-1))\lambda \right)^m z^n$ and $\gamma_i = 0$, for all $i = 1, \ldots, n$, in Theorem 3, we have

**Corollary 3.** ([7])*Let* $\alpha_i \in \mathbb{C}$ *for all* $i = 1, \ldots, n$ *and* $\beta \in \mathbb{C}$ *with*

$$\mathrm{Re}(\beta) \geq \sum_{i=1}^{n} |\alpha_i| > 0$$

*and let* $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i| \, .$$

*If*

$$\left| \frac{z(D_\lambda^{m,\eta} f_i(z))'}{D_\lambda^{m,\eta} f_i(z)} - 1 \right| < 1 \qquad (z \in \mathcal{U}, m \in \mathbb{N}_0)$$

*then the integral operator* $I_\beta^{m,\eta}(f_1, ..., f_n)(z)$ *defined by (7) is analytic and univalent in* $\mathcal{U}$.

Finally, we prove the following theorem.

**Theorem 4.** *Let* $\alpha_i \in \mathbb{C}$ , $M_i \geq 1$ *for all* $i = 1, \ldots, n$ *and* $\beta \in \mathbb{C}$ *with*

$$\operatorname{Re}(\beta) \geq \sum_{i=1}^{n} |\alpha_i| \left( (2 - \mu_i) M_i + 1 \right) > 0. \tag{22}$$

*and let* $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\beta)} \sum_{i=1}^{n} |\alpha_i| \left( (2 - \mu_i) M_i + 1 \right).$$

*If* $f_i \in \mathcal{B}(g_i, \mu_i), 0 \leq \mu_i < 1$ *for all* $i = 1, \ldots, n$ , *and*

$$|(f_i * g_i)(z)| \leq M_i \;\; (z \in \mathcal{U}, i = 1, \ldots, n),$$

*then the integral operator* $I_\beta(f_1, ..., f_n; g_1, ..., g_n)$ *defined by (5) is analytic and univalent in* $\mathcal{U}$.

*Proof.* From (21), it follows that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left[ \frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} - 1 \right].$$

Thus, we have

$$\begin{aligned}
\left|c\left|z\right|^{2\beta}+(1-\left|z\right|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| &= \left|c\left|z\right|^{2\beta}+(\frac{1-\left|z\right|^{2\beta}}{\beta})\sum_{i=1}^{n}\alpha_i\left[\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}-1\right]\right| \\
&\leq |c|+\left|\frac{1-\left|z\right|^{2\beta}}{\beta}\right|\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}\right|+1\right) \\
&\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}\right|\left|\frac{(f_i*g_i)(z)}{z}\right|+1\right) \\
&\leq |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}|\alpha_i|\left(\left|\frac{z^2(f_i*g_i)'(z)}{[(f_i*g_i)(z)]^2}-1\right|M_i+M_i+1\right) \\
&\leq |c|+\frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}|\alpha_i|\left((2-\mu_i)M_i+1\right)\leq 1.
\end{aligned}$$

Applying Lemma 3 for the function $h(z)$, we prove that $I_\beta(f_1,...,f_n;g_1,...,g_n)\in\mathcal{S}$.

Letting $g_1=\cdots=g_n=z+\sum_{n=2}^{\infty}\left(\frac{\Gamma(n+1)\Gamma(2-\eta)}{\Gamma(n+1-\eta)}(1+(n-1))\lambda\right)^m z^n$ and $\mu_i=0,$ for all $i=1,\ldots,n,$ in Theorem 4, we have:

**Corollary 4.**([7])*Let* $\alpha_i\in\mathbb{C}$ , $M_i\geq 1$ *for all* $i=1,\ldots,n$ *and* $\beta\in\mathbb{C}$ *with*

$$\mathrm{Re}(\beta)\geq\sum_{i=1}^{n}|\alpha_i|\left(2M_i+1\right)>0.$$

*and let* $c\in\mathbb{C}$ *be such that*

$$|c|\leq 1-\frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}|\alpha_i|\left(2M_i+1\right).$$

*If*

$$\left|\frac{z^2(D_\lambda^{m,\eta}f_i(z))'}{(D_\lambda^{m,\eta}f_i(z))^2}-1\right|<1\qquad(z\in\mathcal{U},m\in\mathbb{N}_0)$$

*and*

$$\left|D_\lambda^{m,\eta}f_i(z)\right|\leq M_i\ \ (z\in\mathcal{U},i=1,\ldots,n),$$

*then the integral operator* $I_\beta^{m,\eta}(f_1,...,f_n)(z)$ *defined by (7) is analytic and univalent in* $\mathcal{U}$.

**Remark 2.** Taking different choices of $g_1,\cdots,g_n$ as stated in Section 1, the above theorems lead to new sufficient conditions for univalency for the integral operators defined in Remark 1.

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B.A. Frasin
Faculty of Science
Department of Mathematics
Al al-Bayt University
P.O. Box: 130095 Mafraq, Jordan
email: *bafrasin@yahoo.com*